Improving Christofides’ Algorithm for the $s$-$t$ Path TSP

Hyung-Chan An

Joint work with Bobby Kleinberg and David Shmoys
Metric TSP

- **Metric (circuit) TSP**
  - Given a weighted graph $G = (V, E)$ ($c : E \rightarrow \mathbb{R}_+$), find a minimum Hamiltonian circuit
  - Triangle inequality holds
  - Christofides (1976) gave a $3/2$-approximation algorithm

Figure from [Dantzig, Fulkerson, Johnson 1954]
Metric TSP

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  - Triangle inequality holds
  - Christofides (1976) gave a 3/2-approximation algorithm
    - No better performance guarantee known

Figure from [Dantzig, Fulkerson, Johnson 1954]
Metric s-t Path TSP

- Metric *s-t path* TSP
  - Given a weighted graph $G = (V, E)$ ($c : E \rightarrow \mathbb{R}_+$) with endpoints $s, t \in V$, find a minimum s-t Hamiltonian *path*
  - Triangle inequality holds
  - Hoogeveen (1991) showed that Christofides’ algorithm is a $5/3$-approximation algorithm and this bound is tight

Figure from [Dantzig, Fulkerson, Johnson 1954]
Our Main Result

Theorem

*Christofides’ algorithm can be improved to yield a deterministic \( \phi \)-approximation algorithm for the s-t path TSP for an arbitrary metric, where \( \phi = \frac{1+\sqrt{5}}{2} \) is the golden ratio (\( \phi < 1.6181 \)).*
Recent Exciting Improvements

- Recent improvements for **unit-weight graphical metric TSP**
  - Shortest path metric in an underlying unweighted graph
  - Better approximation than Christofides’
    ([Oveis Gharan, Saberi, Singh 2011],
    [Mömke, Svensson 2011], [Mucha 2011],
    [Sebő, Vygen 2012])
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  - Techniques can be successfully applied to both variants

- Our algorithm for the *s-t path* TSP improves Christofides’ for an *arbitrary* metric
  - Can our techniques be extended to the circuit variant?
Can Randomization Beat Christofides?
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- Find minimum span. tree $T_{\text{min}}$
- Augment $T_{\text{min}}$ into a low-cost Eulerian circuit/path
- Transform it into a Hamiltonian circuit/path of no greater cost
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- Choose random span. tree $\mathcal{I}$
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- Asadpour, Goemans, Mądry, Oveis Gharan, Saberi 2010:
  - $O(\log n / \log \log n)$-approx for ATSP

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- $\phi$-approx for s-t path TSP
  - Arbitrary metric
  - Simpler random choice
Christofides’ Algorithm

Christofides’ algorithm

Find a minimum spanning tree $T_{\text{min}}$

Let $T$ be the set of vertices with “wrong” parity of degree:

i.e., $T$ is the set of odd-degree vertices in $T_{\text{min}}$

Find a minimum perfect matching $M$ on $T$

Find an Eulerian circuit of $T_{\text{min}} \cup M$

Shortcut it into a Hamiltonian circuit $H$
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![Graph Image]
Path-variant Christofides’ algorithm

- Path-variant Christofides’ algorithm
  - 5/3-approximation algorithm [Hoogeveen 1991]
  - This bound is tight

Unit-weight graphical metric:
- distance between two vertices defined as shortest distance on this underlying unit-weight graph
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Held-Karp Relaxation

- Held-Karp relaxation
  - \( \delta(S) \) for \( S \subseteq V \) denotes the set of edges in cut \( (S, \bar{S}) \)

Incidence vector \( \chi_F \) of \( F \subseteq E \) is \( (\chi_F)_e := \begin{cases} 1 & \text{if } e \in F \\ 0 & \text{otherwise} \end{cases} \)
Held-Karp Relaxation

- Held-Karp relaxation
  For $G = (V, E)$ and $s, t \in V$,

  \[
  \begin{align*}
  \sum_{e \in \delta(S)} x_e &= 1, \\
  \sum_{v \in V \setminus \{s, t\}} \sum_{e \in \delta(v)} x_e &= 1, \\
  \sum_{e \in \delta(S)} x_e &\geq 2, \\
  \sum_{e \in \delta(S)} x_e &\geq 2, \\
  0 &\leq x_e \leq 1, \\
  x &\in \mathbb{R}^E
  \end{align*}
  \]

  $x \in \mathbb{R}^E$
Held-Karp Relaxation

- Polynomial-time solvable
- Feasible region of this LP is contained in the ST polytope

\[ \text{[Grötschel, Lovász, Schrijver 1981]} \]
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- Held-Karp solution can be written as a convex combination of (incidence vectors of) spanning trees
Held-Karp Relaxation

- Polynomial-time solvable
- Feasible region of this LP is contained in the ST polytope
- Held-Karp solution can be written as a convex combination of (incidence vectors of) spanning trees
- Can find such a decomposition in polynomial time [Grötschel, Lovász, Schrijver 1981]
Our Algorithm

- **Best-of-Many Christofides’ Algorithm**
  - *Compute an optimal solution* $x^\ast$ *to the Held-Karp relaxation*
  - *Rewrite* $x^\ast$ *as a convex comb. of spanning trees* $T_1, \ldots, T_k$
  - For each $T_i$:
    - Let $T_i$ be the set of vertices with “wrong” parity of degree: i.e., $T_i$ is the set of even-degree endpoints and other odd-degree vertices in $T_i$
    - Find a minimum perfect matching $M_i$ on $T_i$
    - Find an $s$-$t$ Eulerian path of $T_i \cup M_i$
    - Shortcut it into an $s$-$t$ Hamiltonian path $H_i$
  - Output the best Hamiltonian path
Randomized Algorithm

- Sampling Christofides’ Algorithm
  - Sample $\mathcal{T}$ by choosing $\mathcal{T}_i$ with probability $\lambda_i$
  
  \[ \chi^* = \sum_{i=1}^{k} \lambda_i \chi_{\mathcal{T}_i} \]
Randomized Algorithm

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- $\mathbb{E}[c(H)] \leq \rho \cdot \text{OPT} \implies$ Best-of-Many Christofides’ Algorithm is $\rho$-approx. algorithm
Randomized Algorithm

- **Sampling Christofides’ Algorithm**
  - Sample $\mathcal{T}$ by choosing $\mathcal{T}_i$ with probability $\lambda_i$
    \[x^* = \sum_{i=1}^{k} \lambda_i x_{\mathcal{T}_i}\]

- $E[c(H)] \leq \rho \cdot \text{OPT} \implies$ Best-of-Many Christofides’ Algorithm is $\rho$-approx. algorithm

- $\Pr[e \in \mathcal{T}] = x_e^*$
Randomized Algorithm

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- $\mathbb{E}[c(H)] \leq \rho \cdot \text{OPT} \implies$
  - Best-of-Many Christofides’ Algorithm is $\rho$-approx. algorithm

- $\Pr[e \in \mathcal{T}] = x^*_e$
  - $\mathbb{E}[c(\mathcal{T})] = \sum_{e \in E} c_e x^*_e = c(x^*)$
  - The rest of the analysis focuses on bounding $c(M)$
Polyhedral Characterization of Matchings

- Polyhedral characterization of matchings on $T$
  (assuming triangle inequality)  [Edmonds, Johnson 1973]

\[
\begin{align*}
\sum_{e \in \delta(S)} y_e &\geq 1, \quad \forall S \subset V, |S \cap T| \text{ odd} \\
y &\in \mathbb{R}^E_+
\end{align*}
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y & \in \mathbb{R}_+^E
\end{align*}
\]

Call a feasible solution a *fractional matching*; its cost upper-bounds $c(M)$
Proof of $5/3$-approximation

- Want: a fractional matching $y$ with $E[c(y)] \leq \frac{2}{3} c(x^*)$
- $x^* :=$ optimal Held-Karp solution
Proof of 5/3-approximation

• Want: a fractional matching $y$ with $E[c(y)] \leq \frac{2}{3} c(x^*)$
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• Take $y := \alpha \chi_T + \beta x^*$ for $\alpha = \beta = \frac{1}{3}$
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\begin{align*}
(Matching) \quad & \left\{ \begin{array}{c}
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(Held-Karp)

\[
\begin{align*}
\sum_{e \in \delta \{s\}} x_e &= 1, \\
\sum_{e \in \delta \{t\}} x_e &= 1, \\
\sum_{e \in \delta \{v\}} x_e &= 2, \quad \forall v \in V \setminus \{s, t\}, \\
\sum_{e \in \delta(S)} x_e &\geq 1, \quad \forall S \subsetneq V, |\{s, t\} \cap S| = 1, \\
\sum_{e \in \delta(S)} x_e &\geq 2, \quad \forall S \subsetneq V, |\{s, t\} \cap S| \neq 1, S \neq \emptyset, \\
0 &\leq x_e \leq 1, \quad \forall e \in E
\end{align*}
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Proof of $5/3$-approximation

- Want: a fractional matching $y$ with $E[c(y)] \leq \frac{2}{3}c(x^*)$
  
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**Lemma**

An s-t cut $(U, \bar{U})$ that is odd w.r.t. $T$ (i.e., $|U \cap T|$ is odd) has at least two tree edges in it
Proof of $5/3$-approximation

- Want: a fractional matching $y$ with $E[c(y)] \leq \frac{2}{3}c(x^*)$
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- $E[c(y)] = \alpha E[c(\chi_T)] + \beta c(x^*) = (\alpha + \beta)c(x^*)$
- $E[c(H)] \leq E[c(T)] + E[c(M)] \leq (1 + \alpha + \beta)c(x^*)$

**Theorem**

*The given algorithm is a $(1 + \alpha + \beta)$-approximation algorithm*
Improvement upon 5/3

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- Perturb $\alpha$ and $\beta$
  - In particular, decrease $\alpha$ by $2\epsilon$ and increase $\beta$ by $\epsilon$
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- $E[c(y)] = (\alpha + \beta)c(x^*)$ decreases by $\epsilon c(x^*)$
- $\alpha + 2\beta$ unchanged; nonseparating cuts remain satisfied
- $T$-odd s-t cuts with small capacity may become violated
  - If violated, by at most $d := O(\epsilon)$
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  - If violated, by at most $d := O(\epsilon)$

**Definition**

For $0 < \tau \leq 1$, a $\tau$-narrow cut $(U, \bar{U})$ is an $s$-$t$ cut with $\sum_{e \in \delta(U)} x^*_e < 1 + \tau$
Improvement upon 5/3

- $\tau$-narrow cuts may be violated when they are $T$-odd

Lemma

For any $\tau$-narrow cut $(U, \bar{U})$, 

$$\Pr[|U \cap T| \text{ odd}] < \tau$$

Proof.

Expected number of tree edges in the cut is 

$$\sum_{e \in \delta(U)} \Pr[e \in T] = \sum_{e \in \delta(U)} x^* e < 1 + \tau \quad (U, \bar{U}) \text{ has at least one tree edge in it}$$

If $(U, \bar{U})$ is odd w.r.t. $T$, it must have another tree edge in it

Lemma

An s-t cut $(U, \bar{U})$ that is odd w.r.t. $T$ (i.e., $|U \cap T|$ is odd) has at least two tree edges in it
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*For any $\tau$-narrow cut $(U, \bar{U})$, Pr[|$U \cap T$| odd] < $\tau$*

**Proof.**

- Expected number of tree edges in the cut is < $1 + \tau$:

\[
\sum_{e \in \delta(U)} \Pr[e \in \mathcal{T}] = \sum_{e \in \delta(U)} x_e^* < 1 + \tau
\]

- $\Pr[e \in \mathcal{T}] = x_e^*$
Improvement upon 5/3

- \(\tau\)-narrow cuts may be violated when they are \(T\)-odd

**Lemma**

*For any \(\tau\)-narrow cut \((U, \bar{U})\), \(\Pr[|U \cap T| \text{ odd}] < \tau\)*

**Proof.**

- Expected number of tree edges in the cut is \(< 1 + \tau\):
  \[
  \sum_{e \in \delta(U)} \Pr[e \in \mathcal{T}] = \sum_{e \in \delta(U)} x_e^* < 1 + \tau
  \]
- \((U, \bar{U})\) has at least one tree edge in it
- If \((U, \bar{U})\) is odd w.r.t. \(T\), it must have another tree edge in it

- \(\Pr[e \in \mathcal{T}] = x_e^*\)

**Lemma**

*An s-t cut \((U, \bar{U})\) that is odd w.r.t. \(T\) (i.e., \(|U \cap T|\) is odd) has at least two tree edges in it*
Improvement upon $5/3$

- $\tau$-narrow cuts may be violated when they are $T$-odd
- This happens with probability smaller than $\tau = O(\epsilon)$
- When this happens, the cut will have deficiency $d = O(\epsilon)$
Improvement upon $5/3$

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Suppose edge sets of $\tau$-narrow cuts were disjoint

$$y := \alpha \chi_T + \beta x^* + r$$

For each $e$, if $e$ is in a $\tau$-narrow cut that is odd w.r.t. $T$, set $r_e := dx_e^*$

Claim $y$ is a fractional matching

Claim $E[c(r)] \leq d\tau c(x^*)$
Improvement upon $5/3$

- $\tau$-narrow cuts are not disjoint
Improvement upon 5/3

- $\tau$-narrow cuts are not disjoint, but “almost” disjoint

Lemma

$\tau$-narrow cuts do not cross: i.e., for $\tau$-narrow cuts $(U, \bar{U})$ and $(W, \bar{W})$ with $s \in U, W$, either $U \subset W$ or $W \subset U$. 

![Diagram showing disjoint and non-disjoint sets](image)
Improvement upon 5/3

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**Lemma**

Each $\tau$-narrow cut has a “representative” edge set of capacity $\geq 1 - \frac{\tau}{2}$, and they are mutually disjoint
The Main Result

Theorem

Best-of-many Christofides’ algorithm is a deterministic $\phi$-approximation algorithm for the s-t path TSP for the general metric, where $\phi = \frac{1+\sqrt{5}}{2} < 1.6181$ is the golden ratio.
Circuit TSP
- Is there a better than 3/2-approximation algorithm?
- Do our techniques extend to the circuit TSP?
Thank you.