Improving Christofides' Algorithm for the s-t Path TSP

Hyung-Chan An

Joint work with Robert Kleinberg and David B. Shmoys

Traveling Salesman Problem

- (Circuit) Traveling Salesman Problem
  - Given a weighted graph $G = (V, E)$ $(c : E \to \mathbb{R}_+)$, find a minimum Hamiltonian circuit

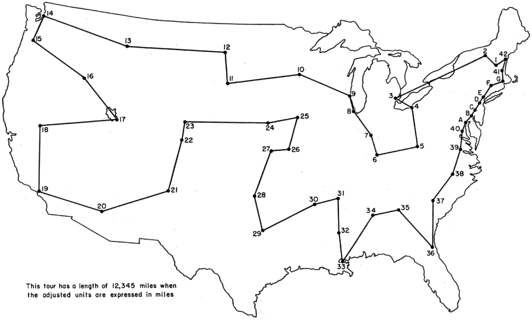

FIG. 16. The optimal tour of 49 cities.

Figure from [Dantzig, Fulkerson, Johnson 1954].

Traveling Salesman Problem

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  - Triangle inequality holds
    or
    Multiple visits to the same vertex allowed
  - NP-hard
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A ρ-approximation algorithm is a poly-time algorithm that produces a solution of cost within ρ times the optimum

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Traveling Salesman Problem

- Metric *s-t path* TSP
  - Given a weighted graph $G = (V, E)$ $(c : E \rightarrow \mathbb{R}_+)$ with *endpoints* $s, t \in V$, find a minimum *s*-*t* Hamiltonian *path*
  - Triangle inequality holds
    or
    Multiple visits to the same vertex allowed
  - NP-hard
  - Hoogeveen (1991) showed that Christofides' algorithm is a $5/3$-approximation algorithm and this bound is tight

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Our Main Result

Theorem

There exists a deterministic ϕ-approximation algorithm for the metric s-t path TSP, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio ($\phi < 1.6181$)

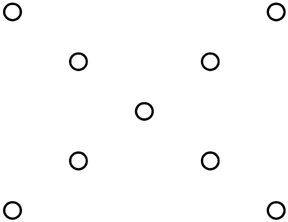
Outline

- Christofides' algorithm
- Linear programming relaxation
- LP-based analysis of Christofides' algorithm
- Path-variant relaxation

- Our algorithm
- Analysis
  - First analysis: proof of $5/3$-approximation
  - Second analysis: first improvement upon $5/3$
  - Last analysis: pushing towards the golden ratio
- Application & open questions

Christofides' Algorithm

- Christofides' algorithm



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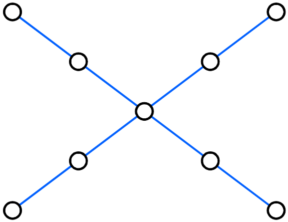

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Graph G has an Eulerian circuit if and only if G is connected and every vertex of G has even degree

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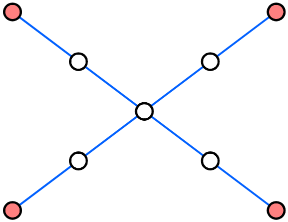

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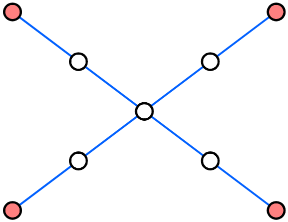

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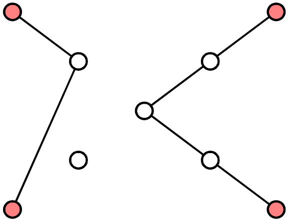

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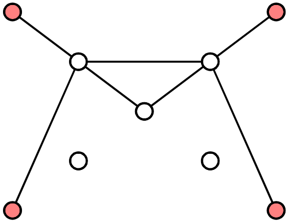

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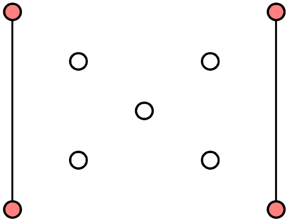

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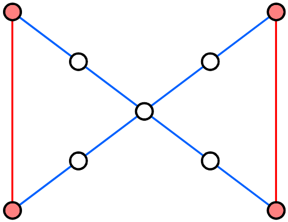

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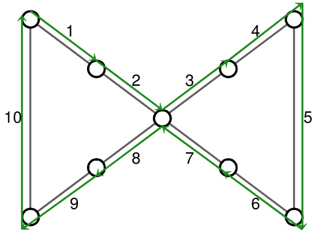

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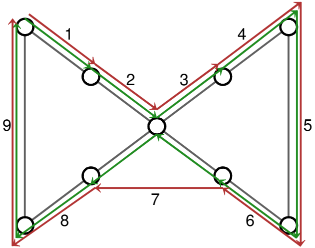

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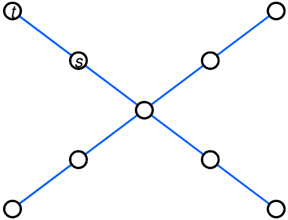

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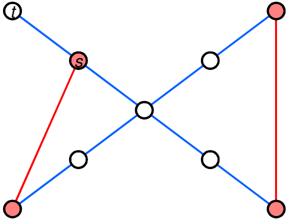

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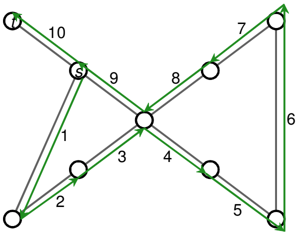

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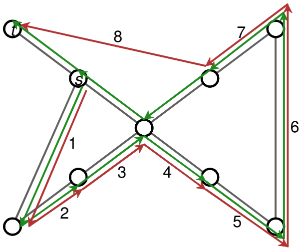

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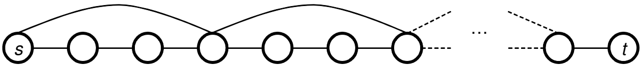

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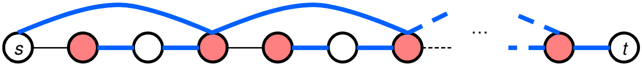

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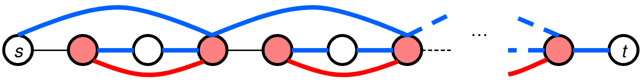

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Recent Exciting Improvements

- Recent improvements for unit-weight graphical metric TSP
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  - Techniques can be successfully applied to both variants
- Our algorithm for the *s*-*t* path TSP improves Christofides' for an *arbitrary* metric
  - Can our techniques be extended to the circuit variant?

LP-based Approximation Algorithms

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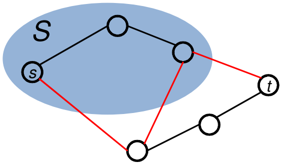
  - $\delta(S)$ for $S \subsetneq V$ denotes the set of edges in cut $(S, \bar{S})$



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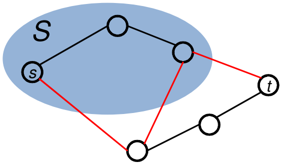


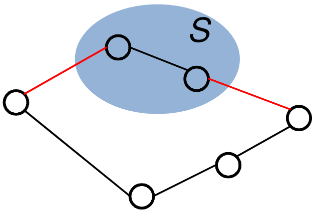
  - For $x, y \in \mathbb{R}_+^E$ and $F \subset E$,
  - $x(y) := \sum_{e \in E} x_e y_e$
  - $x(F) := \sum_{f \in F} x_f$
  - Incidence vector of F is $(\chi_F)_e := \begin{cases} 1 & \text{if } e \in F \\ 0 & \text{otherwise} \end{cases}$

Held-Karp Relaxation

- Held-Karp relaxation (for circuit TSP)
  ([Dantzig, Fulkerson, Johnson 1954], [Held, Karp 1970])

  For $G = (V, E)$,

$$\begin{cases} \sum_{e \in \delta(S)} x_e \geq 2, & \forall S \subsetneq V, S \neq \emptyset \\ \sum_{e \in \delta(\{v\})} x_e = 2, & \forall v \in V \\ x_e \in \{0, 1\} & \forall e \in E \end{cases}$$



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0 \leq x_e \leq 1 & \forall e \in E
\end{cases}
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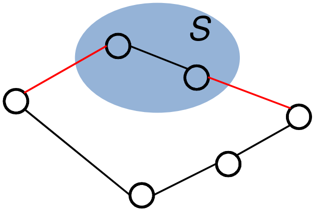
$$x \in \mathbb{R}^E$$



Let x^* be LP optimum;
$c(x^*) \leq c(\mathrm{OPT})$

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  - Any feasible solution to this LP, scaled by $\frac{n-1}{n}$, is in the spanning tree polytope
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  - $c(\mathscr{T}_{\min}) \leq c(\frac{n-1}{n}x^*) \leq c(x^*)$

Polyhedral Characterization of T-joins

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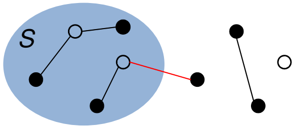
For $T \subset V$, $J \subset E$ is a T-join if the set of odd-degree vertices in $G' = (V, J)$ is T

- Polyhedral characterization of T-joins



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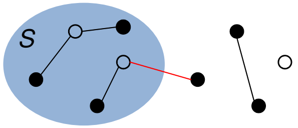

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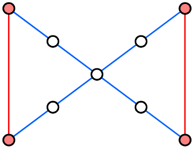
- Call a feasible solution a *fractional T-join*; its cost upper-bounds $c(J)$

LP-based Analysis of Christofides' Algorithm

Theorem (Wolsey 1980)

Christofides' algorithm is a $3/2$*-approximation algorithm*



Proof.

$c(\mathscr{T}_{\min}) \leq c(\frac{n-1}{n}x^*) \leq c(x^*)$

$y^* := \frac{1}{2}x^*$ is a fractional T-join

$$\text{(Held-Karp)} \quad \begin{cases} \sum_{e\in\delta(S)} x_e \geq 2, & \forall S \subsetneq V, S \neq \emptyset \\ \sum_{e\in\delta(\{v\})} x_e = 2, & \forall v \in V \\ 0 \leq x_e \leq 1 & \forall e \in E \end{cases}$$

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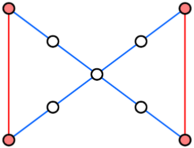

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$c(J) \leq c(y^*) \leq \frac{1}{2}c(x^*)$

$c(H) \leq c(\mathscr{T}_{\min} \cup J) \leq c(x^*) + c(y^*) \leq \frac{3}{2}c(x^*) \leq \frac{3}{2}c(\text{OPT})$ $\square$

Strength of Held-Karp Relaxation

- Integrality gap
  - Worst-case ratio of the integral optimum to the fractional optimum

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- Integrality gap
  - Worst-case ratio of the integral optimum to the fractional optimum
  - $\left[\frac{4}{3}, \frac{3}{2}\right]$; conjectured $\frac{4}{3}$
- Path-case
  - $\left[\frac{3}{2}, \frac{1+\sqrt{5}}{2}\right]$; $\frac{3}{2}$?

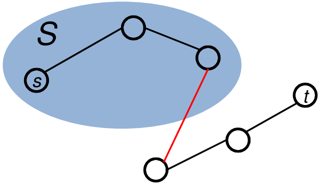
Path-variant Held-Karp Relaxation

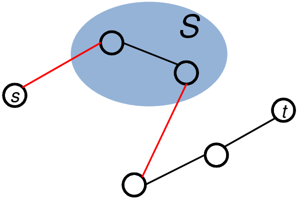
- Path-variant Held-Karp relaxation

  For $G = (V, E)$ and $s, t \in V$,

$$\begin{cases} \sum_{e \in \delta(S)} x_e \geq 1, & \forall S \subsetneq V, |\{s,t\} \cap S| = 1 \\ \sum_{e \in \delta(S)} x_e \geq 2, & \forall S \subsetneq V, |\{s,t\} \cap S| \neq 1, S \neq \emptyset \\ \sum_{e \in \delta(\{s\})} x_e = \sum_{e \in \delta(\{t\})} x_e = 1 & \\ \sum_{e \in \delta(\{v\})} x_e = 2, & \forall v \in V \setminus \{s,t\} \\ 0 \leq x_e \leq 1 & \forall e \in E \end{cases}$$

$x \in \mathbb{R}^E$





Path-variant Held-Karp Relaxation

- Path-variant Held-Karp relaxation
  - Polynomial-time solvable
  - The feasible region of this LP is contained in the spanning tree polytope

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Path-variant Held-Karp Relaxation

- Path-variant Held-Karp relaxation
  - Polynomial-time solvable
  - The feasible region of this LP is contained in the spanning tree polytope
    - A path-variant Held-Karp solution can be written as a convex combination of (incidence vectors of) spanning trees
- Can find such a decomposition in polynomial time [Grötschel, Lovász, Schrijver 1981]
- Try each of these polynomially many spanning trees

Our Algorithm

- Best-of-Many Christofides' Algorithm
  - *Compute an optimal solution x^* to the Held-Karp relaxation*
  - *Rewrite x^* as a convex comb. of spanning trees $\mathscr{T}_1, \ldots, \mathscr{T}_k$*
  - For each $\mathscr{T}_i$:
    - Let T_i be the set of vertices with "wrong" parity of degree: i.e., T_i is the set of even-degree endpoints and other odd-degree vertices in $\mathscr{T}_i$
    - Find a minimum T_i-join J_i
    - Find an s-t Eulerian path of $\mathscr{T}_i \cup J_i$
    - Shortcut it into an s-t Hamiltonian path H_i
  - Output the best Hamiltonian path

Randomized Algorithm

- Randomized algorithm for notational convenience

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  - Compute an optimal solution x^* to the Held-Karp relaxation
  - Rewrite x^* as a convex comb. of spanning trees $\mathscr{T}_1, \ldots, \mathscr{T}_k$: $x^* = \sum_{i=1}^{k} \lambda_i \chi_{\mathscr{T}_i}, \sum_{i=1}^{k} \lambda_i = 1$
  - *Sample $\mathscr{T}$ by choosing $\mathscr{T}_i$ with probability λ_i*
  - Let T be the set of vertices with "wrong" parity of degree: i.e., T is the set of even-degree endpoints and other odd-degree vertices in $\mathscr{T}$
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  - Find an s-t Eulerian path of $\mathscr{T} \cup J$
  - Shortcut it into an s-t Hamiltonian path H
- $\mathrm{E}[c(H)] \leq \rho \cdot \mathrm{OPT} \implies$
  Best-of-Many Christofides' Algorithm is ρ-approx. algorithm

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  - Find an s-t Eulerian path of $\mathscr{T} \cup J$
  - Shortcut it into an s-t Hamiltonian path H
- $\Pr[e \in \mathscr{T}] = x^*_e$
  - $\mathrm{E}[c(\mathscr{T})] = \sum_{e \in E} c_e x^*_e = c(x^*)$
  - The rest of the analysis focuses on bounding $c(J)$

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Lemma $\mathrm{E}[c(\mathscr{T})] = \sum_{e \in E} c_e x_e^* = c(x^*)$
Lemma $\mathrm{E}[c(J)] \leq \bigstar \cdot c(x^*)$
Corollary $\mathrm{E}[c(H)] \leq \mathrm{E}[c(\mathscr{T} \cup J)] \leq (1 + \bigstar) c(x^*)$

Proof of $5/3$-approximation

- Want: a fractional T-join y with $\mathrm{E}[c(y)] \leq \frac{2}{3}c(x^*)$
  $x^* :=$ optimal path-variant Held-Karp solution

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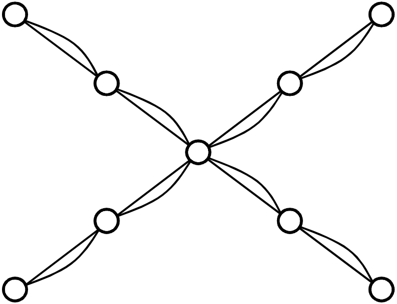
  - Well-known 2-approximation algorithm can be considered as using MST as a fractional T-join



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- Circuit case
  - Well-known 2-approximation algorithm can be considered as using MST as a fractional T-join
  - Christofides' algorithm uses half the (circuit-variant) Held-Karp solution [Wolsey 1980]

Proof of $5/3$-approximation

- Want: a fractional T-join y with $\mathrm{E}[c(y)] \le \frac{2}{3} c(x^*)$
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- Is βx^* a fractional T-join for some constant β?

$$
\text{(Held-Karp)} \qquad
\begin{cases}
\displaystyle\sum_{e \in \delta(S)} x_e \ge 1, & \forall S \subsetneq V, |\{s,t\} \cap S| = 1 \\
\displaystyle\sum_{e \in \delta(S)} x_e \ge 2, & \forall S \subsetneq V, |\{s,t\} \cap S| \ne 1, S \ne \emptyset \\
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0 \le x_e \le 1 & \forall e \in E
\end{cases}
$$

$$
\text{(T-join)} \qquad
\begin{cases}
\displaystyle\sum_{e \in \delta(S)} y_e \ge 1, \quad \forall S \subset V, |S \cap T| \text{ odd} \\
y \in \mathbb{R}_+^E
\end{cases}
$$

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  $x^* :=$ optimal path-variant Held-Karp solution
- Is βx^* a fractional T-join for some constant β?
  - Yes, for $\beta = 1$.
    The present algorithm is a 2-approximation algorithm:
    $\mathrm{E}[c(J)] \le \mathrm{E}[c(\beta x^*)] = \beta c(x^*)$

	x^*
LB on s-t cut capacities	1
LB on nonseparating cut capacities	2

Proof of $5/3$-approximation

- Want: a fractional T-join y with $\mathrm{E}[c(y)] \leq \frac{2}{3}c(x^*)$.
- Is βx^* a fractional T-join for some β? Yes, for $\beta = 1$.
- How about $\alpha\chi_{\mathscr{T}}$?

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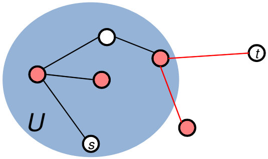
  - *s-t* cuts do have some slack in this case

Lemma

An s-t cut $(U, \bar{U})$ that is odd w.r.t. T (i.e., $|U \cap T|$ is odd) has at least two edges in it.



	$\chi_{\mathscr{T}}$	x^*
LB on *T-odd s-t* cut capacities	2	1
LB on nonseparating cut capacities	1	2

Proof of $5/3$-approximation

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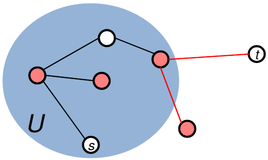

Proof. U contains exactly one of s and t $\Rightarrow$ U has even number of odd-degree vertices

#edges in $\delta(U)$
$= \sum_{v \in U}$ degree of $v - 2 \cdot$ (#edges within U) $\square$

	$\chi_{\mathscr{T}}$	x^*
LB on *T-odd* s-t cut capacities	2	1
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- Yes, for $\alpha = 1$.
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LB on *T-odd* s-t cut capacities	2	1
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Proof of $5/3$-approximation

	$\chi_{\mathscr{T}}$	x^*
LB on T-odd s-t cut capacities	2	1
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Proof of $5/3$-approximation

	$\chi_{\mathscr{T}}$	x^*	y
LB on T-odd s-t cut capacities	2	1	$2\alpha + \beta$
LB on nonseparating cut capacities	1	2	$\alpha + 2\beta$

- $y := \alpha\chi_{\mathscr{T}} + \beta x^*$

Proof of 5/3-approximation

	$\chi_{\mathscr{T}}$	x^*	y
LB on T-odd s-t cut capacities	2	1	$2\alpha + \beta = 1$
LB on nonseparating cut capacities	1	2	$\alpha + 2\beta = 1$

- $y := \alpha\chi_{\mathscr{T}} + \beta x^*$
  - Choose $\alpha = \beta = \frac{1}{3}$
  - The present algorithm is a 5/3-approximation algorithm: $E[c(J)] \leq E[c(y)] = (\alpha + \beta)c(x^*) = \frac{2}{3}c(x^*)$

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- Analysis also works for the original path-variant Christofides' algorithm

First improvement upon $5/3$

	$\chi_{\mathscr{T}}$	x^*	y
LB on T-odd s-t cut capacities	2	1	$2\alpha + \beta = 1$
LB on nonseparating cut capacities	1	2	$\alpha + 2\beta = 1$

- Perturb α and β

First improvement upon $5/3$

	$\chi_{\mathscr{T}}$	x^*	y
LB on T-odd s-t cut capacities	2	1	$2\alpha + \beta =$ 0.95
LB on nonseparating cut capacities	1	2	$\alpha + 2\beta = 1$

- Perturb α and β
  - In particular, decrease α by 2ϵ and increase β by ϵ: will choose $\alpha = 0.30$ and $\beta = 0.35$ later
- $E[c(y)] = (\alpha + \beta)c(x^*)$ decreases by $\epsilon c(x^*)$
- $\alpha + 2\beta$ unchanged; only s-t cuts may be violated by at most $1 - (2\alpha + \beta) =: d$. $d = 0.05$

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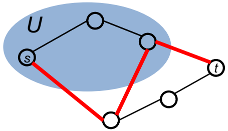
- s-t cuts $(U, \bar{U})$ with large capacity in HK solution are safe: $2\alpha + \beta x^*(\delta(U))$ still large



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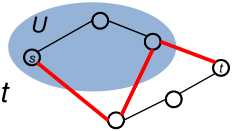

Definition

For $0 < \tau \leq 1$, a τ-narrow cut $(U, \bar{U})$ is an s-t cut with $x^*(\delta(U)) < 1 + \tau$

- $2\alpha + \beta(1 + \tau) = 1$: $\tau = \frac{1}{7}$

First improvement upon $5/3$

- s-t cuts $(U, \bar{U})$ with $x^*(\delta(U)) = 1$ are safe

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Lemma

An s-t cut $(U, \bar{U})$ *that is odd w.r.t.* T *(i.e.,* $|U \cap T|$ *is odd) has at least two edges in it*

Corollary

Each s-t cut $(U, \bar{U})$ *with* $x^*(\delta(U)) = 1$ *is never odd w.r.t.* T

$$\begin{cases} \sum_{e \in \delta(S)} y_e \geq 1, & \forall S \subset V, |S \cap T| \text{ odd} \\ y \in \mathbb{R}_+^E \end{cases}$$

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Corollary

Each s-t cut $(U, \bar{U})$ with $x^*(\delta(U)) = 1$ is never odd w.r.t. T

Proof.

Expected number of tree edges in the cut is equal to $x^*(\delta(U))$:

$$\mathrm{E}[|\delta(U) \cap \mathscr{T}|] = \sum_{e \in \delta(U)} \Pr[e \in \mathscr{T}] = \sum_{e \in \delta(U)} x^*_e = 1$$

So $|\delta(U) \cap \mathscr{T}|$ is identically 1. $\square$

First improvement upon $5/3$

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For any τ-narrow cut $(U, \bar{U})$, $\Pr[|U \cap T| \text{ odd}] < \tau$

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For any τ-narrow cut $(U, \bar{U})$, $\Pr[|U \cap T| \text{ odd}] < \tau$

Proof.

- $(U, \bar{U})$ has at least one tree edge in it
- If $(U, \bar{U})$ is odd w.r.t. T, it must have another tree edge in it
- Expected number of tree edges in the cut is $< 1 + \tau$

$\Pr[|U \cap T| \text{ odd}] \leq x^*(\delta(U)) - 1 < \tau$ $\square$

First improvement upon $5/3$

- Nonseparating cuts and *s*-*t* cuts with high capacities are safe
- For τ-narrow cuts,
  - deficiency is at most $d := 1 - (2\alpha + \beta) = 0.05$
  - probability that the cut is odd w.r.t. T is at most $\tau = \frac{1}{7}$

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- *Suppose* edge sets of τ-narrow cuts were disjoint
- For each τ-narrow cut $(U, \bar{U})$, define "correction vector" f_U defined as the Held-Karp solution restricted to $\delta(U)$

$$(f_U)_e = \begin{cases} x_e^* & \text{if } e \in \delta(U) \\ 0 & \text{otherwise} \end{cases}$$

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- For each τ-narrow cut $(U, \bar{U})$, define "correction vector" f_U defined as the Held-Karp solution restricted to $\delta(U)$
- $y := \alpha\chi_{\mathscr{T}} + \beta x^* + \sum_{U:(U,\bar{U}) \text{ is } \tau-\text{narrow}, |U \cap T| \text{ odd}} d \cdot f_U$

$$\begin{aligned}
& \mathrm{E}\left[c(\textstyle\sum_{U:(U,\bar{U}) \text{ is } \tau-\text{narrow}, |U \cap T| \text{ odd}} d \cdot f_U)\right] \\
\leq \quad & c\left(\textstyle\sum_{U:(U,\bar{U}) \text{ is } \tau-\text{narrow}} \Pr[|U \cap T| \text{ odd}] \cdot d \cdot f_U\right) \\
\leq \quad & d\tau c\left(\textstyle\sum_{U:(U,\bar{U}) \text{ is } \tau-\text{narrow}} f_U\right) \leq d\tau c(x^*)
\end{aligned}$$

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$$\begin{aligned} & \mathrm{E}\left[c\left(\sum_{U:(U,\bar{U}) \text{ is } \tau-\text{narrow},|U\cap T| \text{ odd}} d \cdot f_U\right)\right] \\ \leq\ & c\left(\sum_{U:(U,\bar{U}) \text{ is } \tau-\text{narrow}} \Pr[|U \cap T| \text{ odd}] \cdot d \cdot f_U\right) \\ \leq\ & d\tau c\left(\sum_{U:(U,\bar{U}) \text{ is } \tau-\text{narrow}} f_U\right) \leq d\tau c(x^*) \end{aligned}$$

- The present algorithm is a 1.6572-approximation algorithm *if τ-narrow cuts were disjoint*: $\mathrm{E}[c(y)] \leq (\alpha + \beta + d\tau)c(x^*)$

First improvement upon $5/3$

- τ-narrow cuts are not disjoint

First improvement upon $5/3$

- τ-narrow cuts are not disjoint, but “almost” disjoint

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τ-narrow cuts do not cross: i.e., for τ-narrow cuts $(U, \bar{U})$ and $(W, \bar{W})$ with $s \in U, W$, either $U \subset W$ or $W \subset U$.

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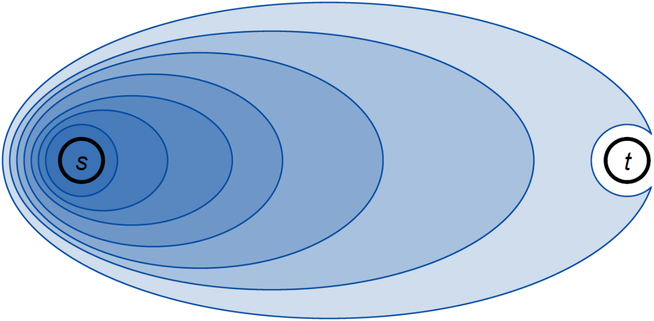

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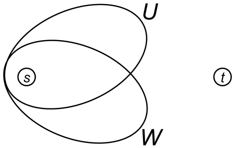

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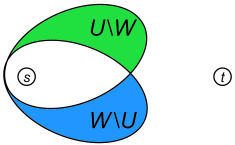


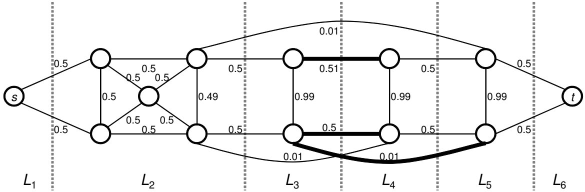
$\square$

First improvement upon $5/3$

Corollary

There exists a partition $L_1, \dots, L_\ell$ of V such that

- $L_1 = \{s\}$, $L_\ell = \{t\}$, *and*
- $\{U | (U, \bar{U})$ *is* τ*-narrow,* $s \in U\} = \{U_i | 1 \le i < \ell\}$, *where* $U_i := \cup_{k=1}^{i} L_k$



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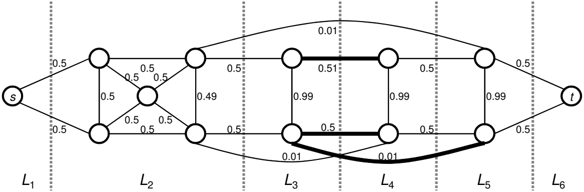

Thick edges show F_3

- We choose "representative edge set" $F_i := E(L_i, L_{\geq i+1})$ for each $\delta(U_i)$. We claim:
  - F_i's are disjoint
  - F_i has large capacity

First improvement upon $5/3$

Lemma

$x^*(F_i) \geq 1 - \frac{\tau}{2}$

Proof.

$A := x^*(E(L_{\leq i-1}, L_i))$, $B := x^*(E(L_i, L_{\geq i+1}))$,
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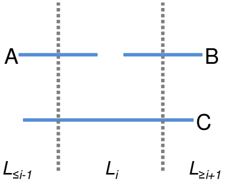

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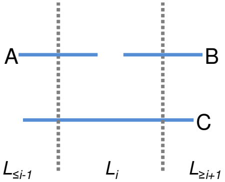

$$B + A \quad \geq \quad 2$$

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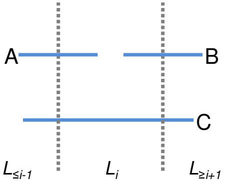

$$
\begin{aligned}
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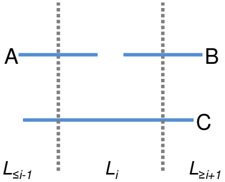

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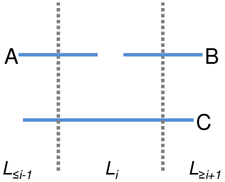

$$\begin{aligned} B + A &\geq 2 \\ B + C &\geq 1 \\ 1 + \tau &> A + C \\ 2B &> 2 - \tau \end{aligned}$$

$$x^*(F_i) = B > 1 - \frac{\tau}{2}$$

$\square$

First improvement upon $5/3$

- τ-narrow cuts are the only cuts that may potentially be violated
- For τ-narrow cuts,
  - deficiency is at most $d := 1 - (2\alpha + \beta) = 0.05$
  - probability that the cut is odd w.r.t. T is at most $\tau = \frac{1}{7}$
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- $\mathrm{E}[c(y)] \leq (\alpha + \beta + d\tau)c(x^*)$
  $\mathrm{E}[c(y)] \leq (\alpha + \beta + \frac{d\tau}{1-\frac{\tau}{2}})c(x^*) \leq 0.6577c(x^*)$
- The present algorithm is a 1.6577-approximation algorithm

Tighter anlysis

- Deficiency and the probability that a τ-narrow cut is odd w.r.t. T were separately bounded
- Write them as a function of the cut capacity and simultaneously optimize
- $x^*(F_i) > 1 - \frac{\tau}{2} + \frac{x^*(\delta(U_i)) - 1}{2}$
- $\frac{9-\sqrt{33}}{2}$-approximation algorithm ($\frac{9-\sqrt{33}}{2} < 1.6278$)

Pushing towards $\frac{1+\sqrt{5}}{2}$ ($\frac{1+\sqrt{5}}{2} < 1.6181$)

Pushing towards $\frac{1+\sqrt{5}}{2}$ ($\frac{1+\sqrt{5}}{2} < 1.6181$)

- Key properties of the correction vectors used in the analysis
  - f_{U_i}'s are nonnegative
  - $\sum_i f_{U_i} \leq x^*$
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First improvement upon $5/3$

- Nonseparating cuts and *s*-*t* cuts with high capacities are safe
- For τ-narrow cuts,
  - deficiency is at most $d := 1 - (2\alpha + \beta) = 0.05$
  - probability that the cut is odd w.r.t. T is at most $\tau = \frac{1}{7}$
- *Suppose* edge sets of τ-narrow cuts were disjoint
- For each τ-narrow cut $(U, \bar{U})$, define "correction vector" f_U defined as the Held-Karp solution restricted to $\delta(U)$
- $y := \alpha\chi_{\mathscr{T}} + \beta x^* + \sum_{U:(U,\bar{U}) \text{ is } \tau-\text{narrow}, |U \cap T| \text{ odd}} d \cdot f_U$

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\leq\ & c\left(\textstyle\sum_{U:(U,\bar{U}) \text{ is } \tau-\text{narrow}} \Pr[|U \cap T| \text{ odd}] \cdot d \cdot f_U\right) \\
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There exists a set of vectors $\{\hat{f}^*_{U_i}\}_{i=1}^{\ell-1}$ *satisfying:*

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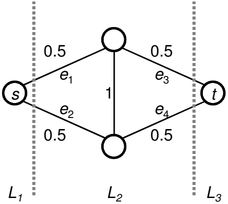
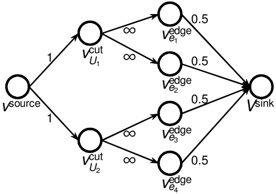
Proof. Consider an auxiliary flow network





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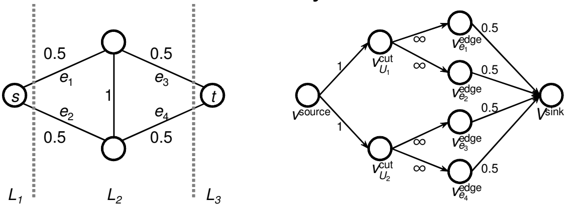
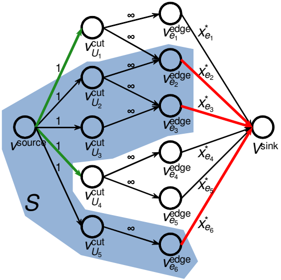


- We claim the maximum flow value on this network is $\ell - 1$
  A maximum flow saturates all the edges from v^{source} to v^{cut}_U
- Define $(\hat{f}^*_U)_e$ as the flow from v^{cut}_U to v^{edge}_e

Pushing towards $\frac{1+\sqrt{5}}{2}$ ($\frac{1+\sqrt{5}}{2} < 1.6181$)



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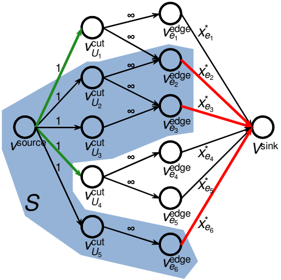

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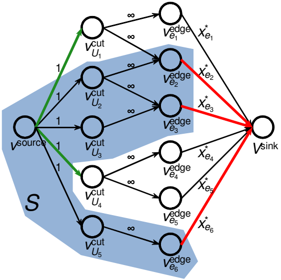

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- Want: if k of the τ-narrow cuts are in S, the edges in any of these k τ-narrow cuts have total Held-Karp value $\geq k$

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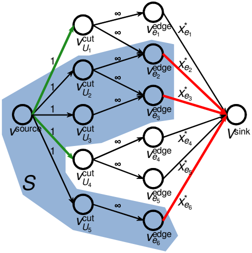

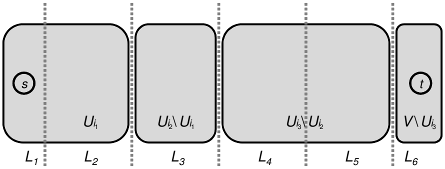

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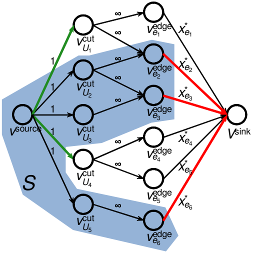

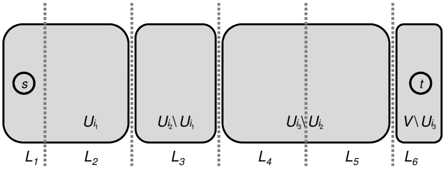

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$$\begin{aligned} & \textstyle\sum_{e:\exists v^{\text{cut}}_U \in S\ e \in \delta(U)} x^*_e \\ = \quad & \tfrac{1}{2}\left[x^*(\delta(U_{i_1})) + \textstyle\sum_{j=2}^{k} x^*(\delta(U_{i_j} \setminus U_{i_{j-1}})) + x^*(\delta(V \setminus U_{i_k}))\right] \\ \geq \quad & \tfrac{1}{2}\left[1 + 2(k-1) + 1\right] = k \end{aligned}$$

$\square$

The Main Result

$$y := \alpha\chi_{\mathscr{T}} + \beta x^* + \sum_{i:|U_i \cap T| \text{ is odd},\ 1 \le i < \ell} \left[1 - \{2\alpha + \beta x^*(\delta(U_i))\}\right] \hat{f}^*_{U_i}$$

for $\alpha = 1 - \frac{2}{\sqrt{5}}$ and $\beta = \frac{1}{\sqrt{5}}$ yields the following:

Theorem

Best-of-many Christofides' algorithm is a deterministic ϕ-approximation algorithm for the s-t path TSP for the general metric, where $\phi = \frac{1+\sqrt{5}}{2} < 1.6181$ is the golden ratio

Applications & open questions

Applications & open questions

- Unit-weight graphical metric case
  - [Oveis Gharan, Saberi, Singh 2011], [Mömke, Svensson 2011], [Mucha 2011]
  - Algorithmic use of τ-narrow cuts
  - A 1.5780-approximation algorithm

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- Prize-collecting *s*-*t* path problem
  - Given a metric cost and vertex prize defined on every vertex, find an *s*-*t* path that minimizes the sum of the path cost and the total prize "missed"

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    - Given a metric cost and vertex prize defined on every vertex, find an s-t path that minimizes the sum of the path cost and the total prize "missed"
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    - A 1.9535-approximation algorithm

Applications & open questions

- Open questions
  - Improve the performance guarantee?
  - Do our techniques extend to the circuit TSP?

Thank you.