Making Three out of Two: Three-Way Online Correlated Selection

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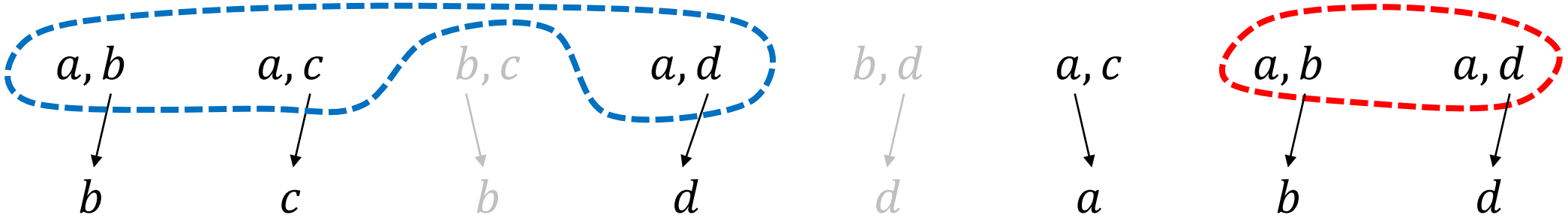
Two-Way Online Correlated Selection (OCS)

- Given a sequence of pairs arriving one-by-one at each timestep, choose an element from each pair irrevocably
- **Definition** (Two-way γ-OCS) [Fahrbach et al.]
  - For any element a and a set of disjoint consecutive subsequences containing a of length $k_1, \cdots, k_\ell$,
    $P[a \text{ never chosen from subseqs}] \leq \prod_{i=1}^{\ell} (1/2)^{k_i} (1-\gamma)^{(k_i-1)_+}$

    $\max(k_i - 1, 0)$
- Negative Correlation



Two-Way Online Correlated Selection (OCS)

- Invented by Fahrbach, Huang, Tao, Zadimoghaddam (FOCS 2020) to tackle the edge-weighted online bipartite matching problem
- Introduces negative correlation to online algorithms
- Fahrbach et al. present
  - 1/16-OCS $\leq \prod_{i=1}^{\ell}(1/2)^{k_i}(1-1/16)^{(k_i-1)_+}$
  - 0.1099-OCS $\leq \prod_{i=1}^{\ell}(1/2)^{k_i}(1-0.1099)^{(k_i-1)_+}$

Open Question

- Can we obtain a nontrivial >2-way OCS?

Yes. We give a three-way OCS.

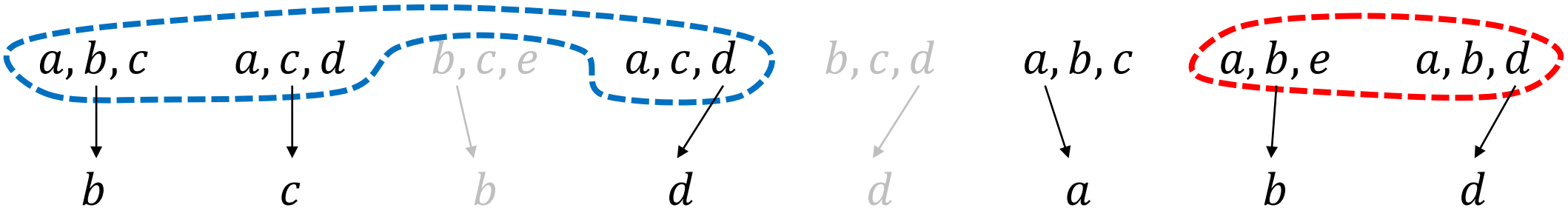
Independent Work

- Blanc and Charikar (FOCS 2021)
  - (F, m)-OCS
  - Continuous-OCS
- Gao et al. (FOCS 2021)
  - Automata-based two-way 0.167-OCS
  - Multiway semi-OCS

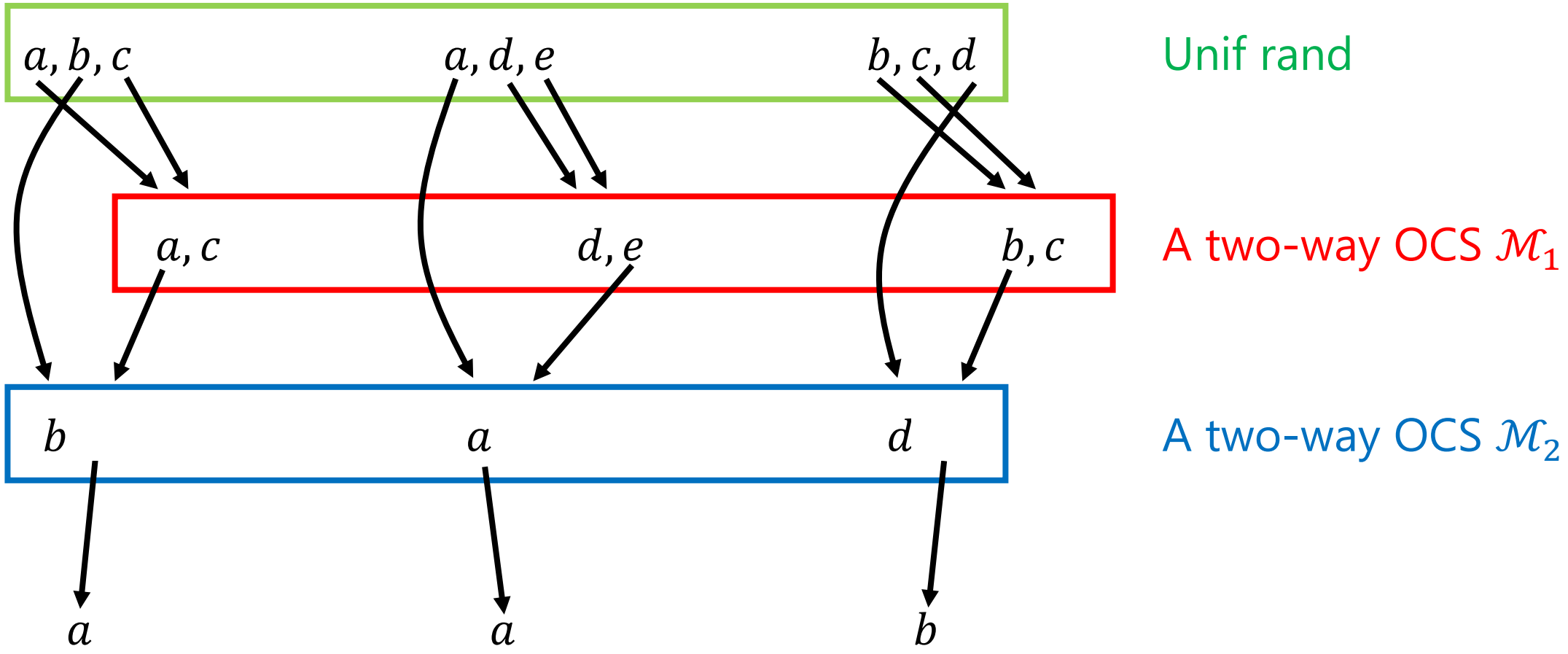
Main Result: Three-Way OCS

- Given a sequence of **triples** arriving one-by-one at each time, choose an element from each **triple** irrevocably
- For any element a and a set of disjoint consecutive subseqs containing a of length $k_1, \cdots, k_\ell$,

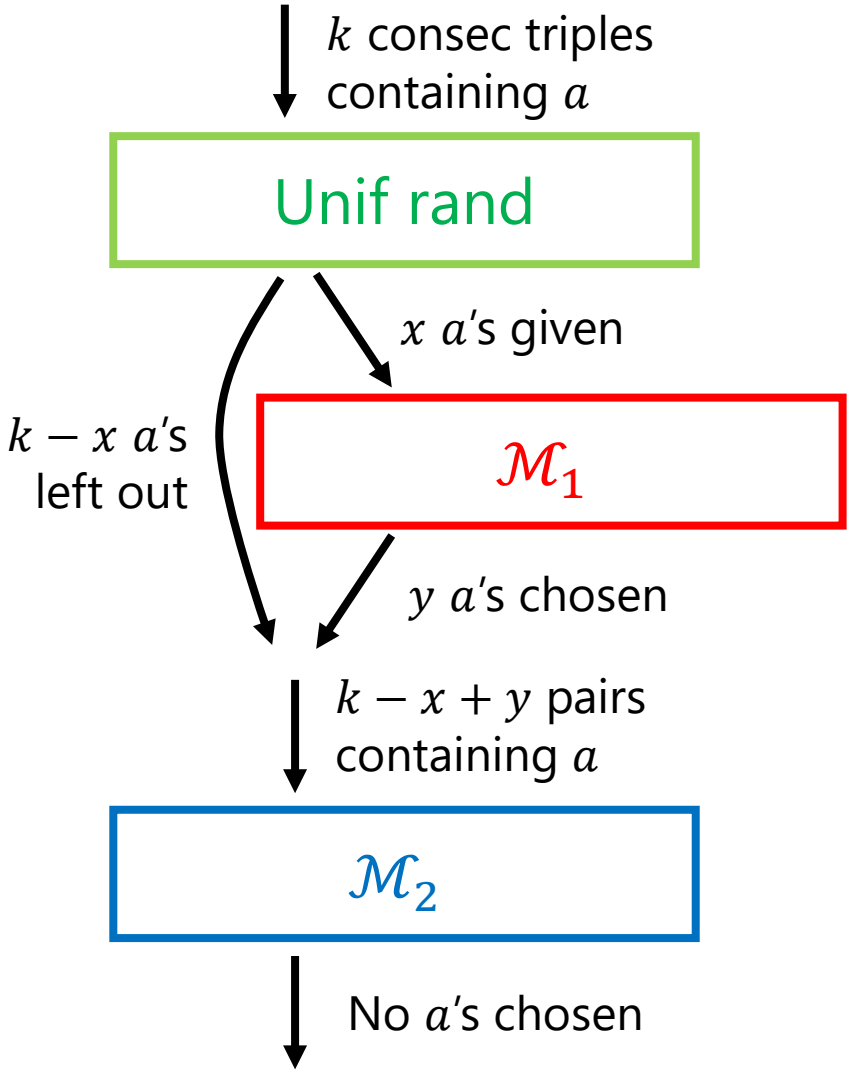
$$P[a \text{ never chosen from subseqs}]$$
$$\leq \prod_{i=1}^{\ell} (2/3)^{k_i} (1-\delta_1)^{(k_i-1)_+} (1-\delta_2)^{(k_i-2)_+}$$



Our Algorithm

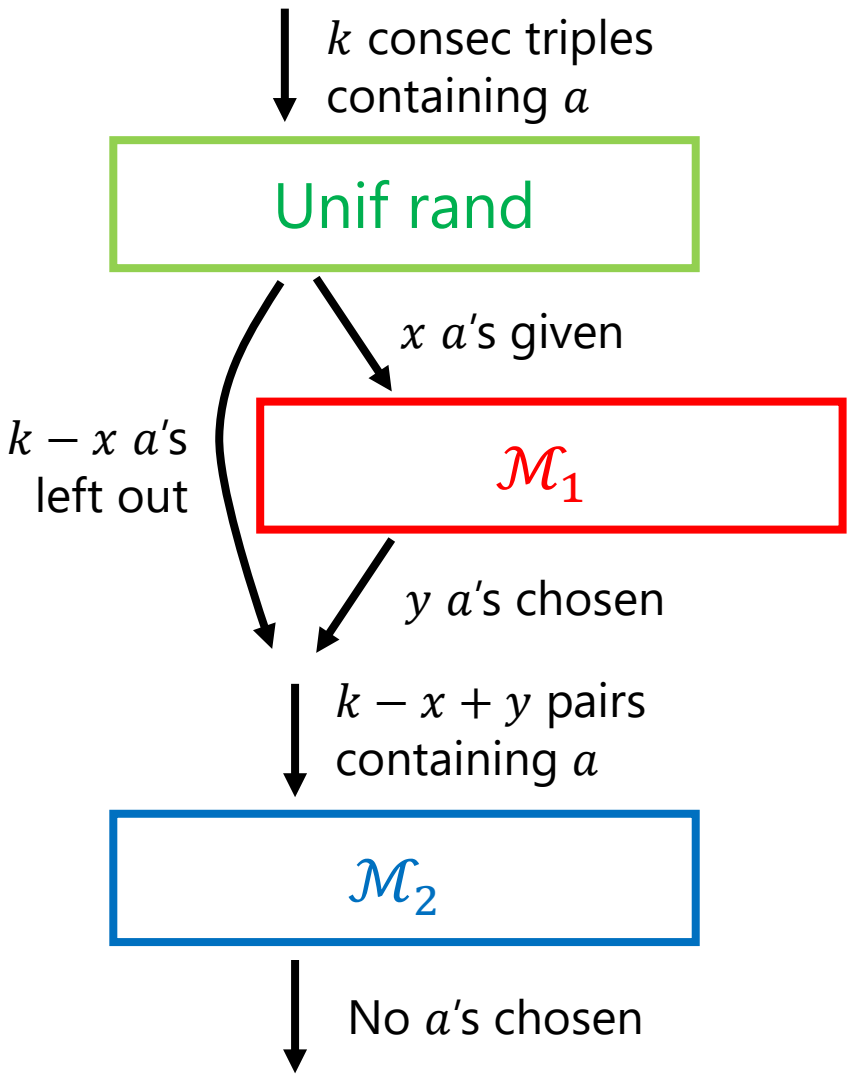


Special Case: A Single Subsequence



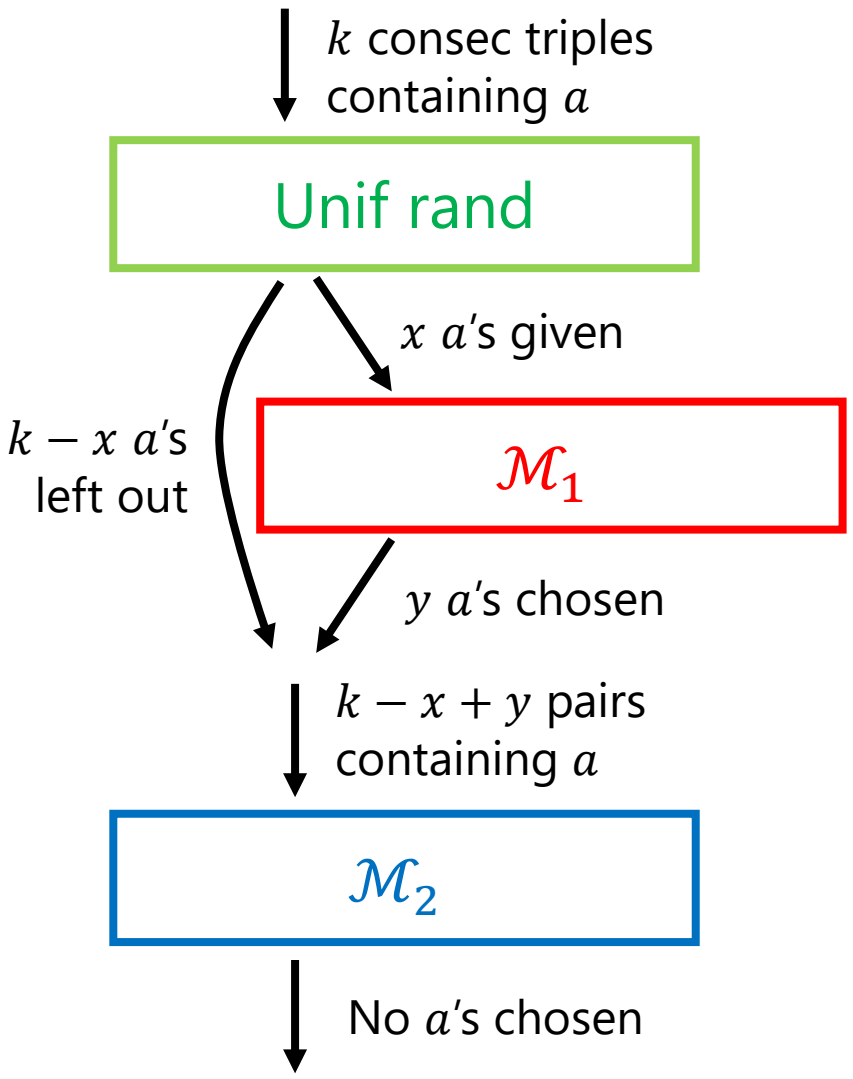
- $P[a \text{ never chosen out of } k \text{ triples}]$
  $\leq \sum_{x=0}^{k} P[x\ a'\text{s given to } \mathcal{M}_1 \text{ out of } k] \cdot$
  $\left[\sum_{y=0}^{x} P[y\ a'\text{s chosen by } \mathcal{M}_1 \text{ out of } x] \cdot\right.$
  $\left.P[\text{No } a's \text{ chosen by } \mathcal{M}_2 \text{ out of } k - x + y]\right]$

Special Case: A Single Subsequence



- $P[a \text{ never chosen out of } k \text{ triples}]$
  $\leq \sum_{x=0}^{k} \mathrm{binom}(k, 2/3\,; x) \cdot$
  $\left[\sum_{y=0}^{x} P[y\ a'\text{s chosen by } \mathcal{M}_1 \text{ out of } x] \cdot\right.$
  $\left.(1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)_+}\right]$

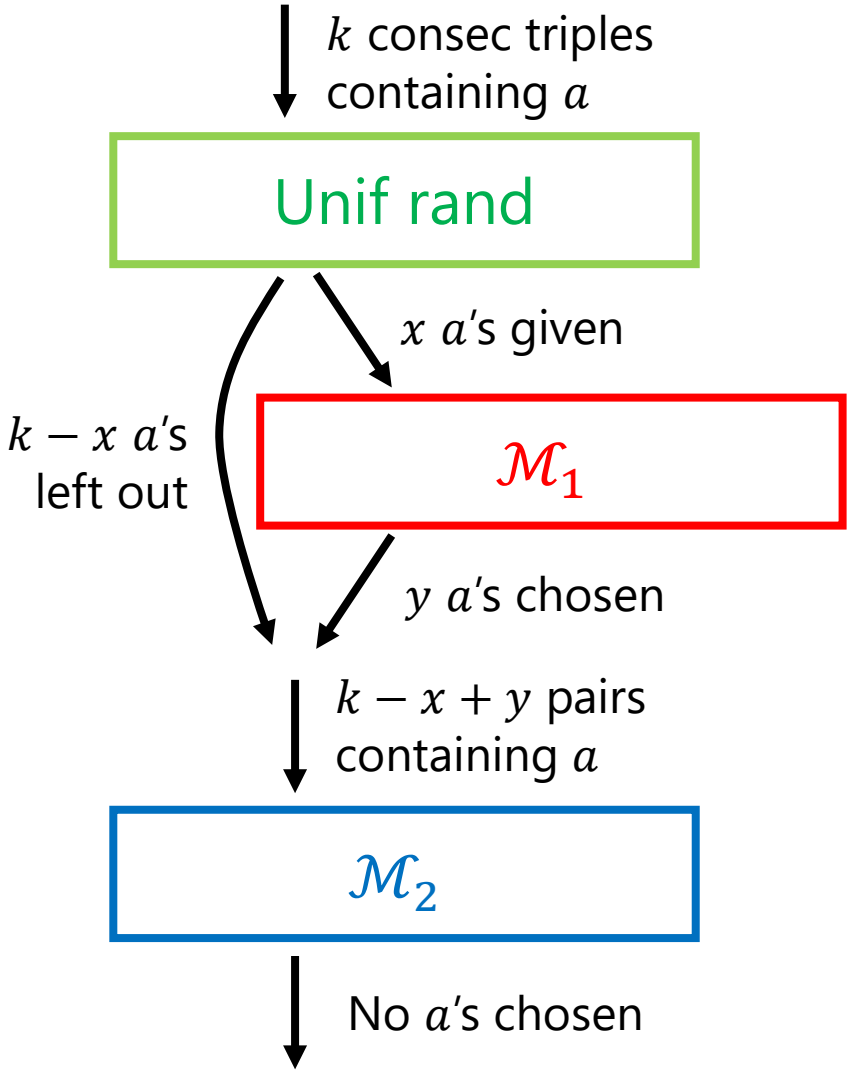
Major Difficulty in Analysis



- $P[a \text{ never chosen out of } k \text{ triples}]$
  $\cancel{\leq \sum_{x=0}^{k} \mathrm{binom}(k, 2/3\,; x) \cdot \left[\sum_{y=0}^{x} P[y\ a\text{'s chosen by } \mathcal{M}_1 \text{ out of } x] \cdot (1/2)^{k-x+y} (1-\gamma)^{(k-x+y-1)_+}\right]}$

- $P[y\ a\text{'s chosen by } \mathcal{M}_1 \text{ out of } x]$
  - was not studied by previous analyses
  - depends on the actual input, not on x
  - does not have a closed-form formula

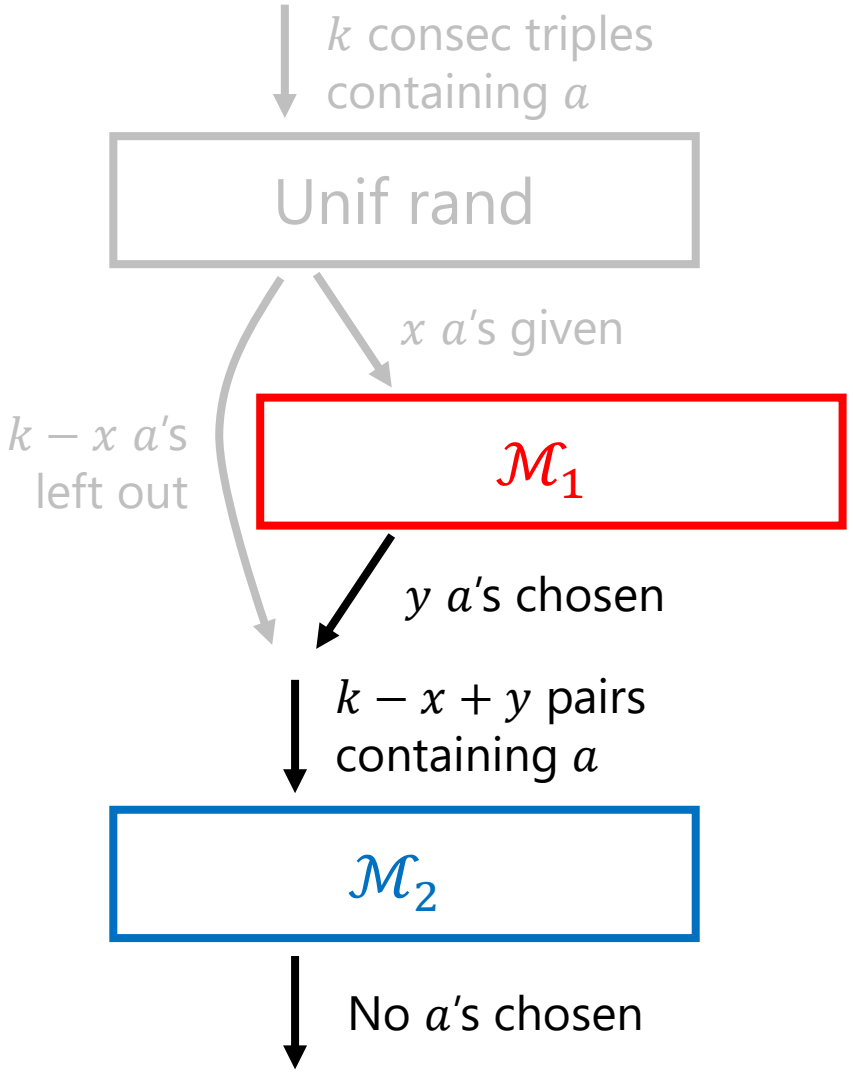
Main Idea of Analysis



- $P[a \text{ never chosen out of } k \text{ triples}]$
  $\leq \sum_{x=0}^{k} \mathrm{binom}(k, 2/3\,; x) \cdot$
  $\left[\sum_{y=0}^{x} p^*(x,y) \cdot\right.$
  $\left.(1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)_+}\right]$

- Construct a "surrogate" distribution $p^*(x,y)$
  - depending only on x
  - yielding a closed-form formula

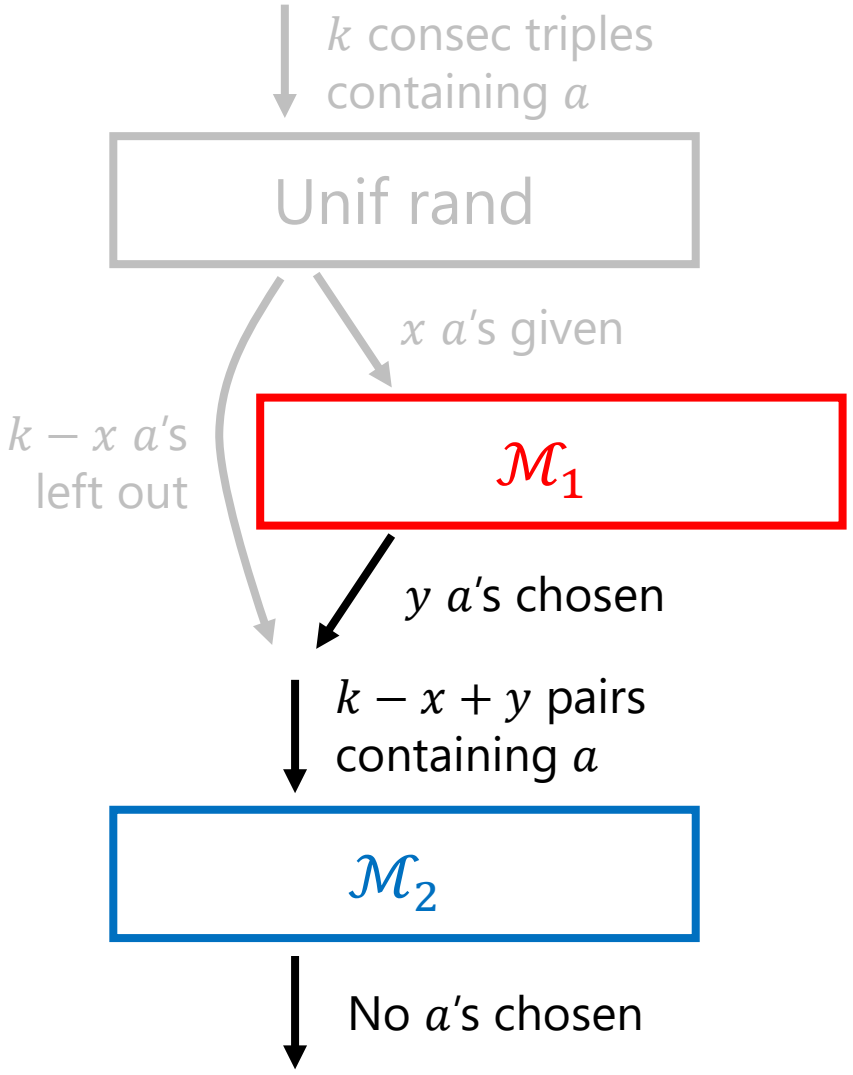
Fixing the Input to $\mathcal{M}_1$



- $P[\, a$ never chosen out of k triples | Unif rand $]$
  $\leq \sum_{y=0}^{x} p(y) \times (1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)_+}$

- Fix the input to $\mathcal{M}_1$ by conditioning Unif rand
- $x :=$ #a's given to $\mathcal{M}_1$ in the conditioned input
- $p(y) := P[y\ a\text{'s chosen by } \mathcal{M}_1$ | Unif rand$]$

Property of Surrogate Distribution



- $P[\, a$ never chosen out of k triples | Unif rand $]$
  $\leq \mathbb{E}_{y\sim p(\cdot)}\left[(1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)_+}\right]$
  $\leq \mathbb{E}_{y\sim p^*(x,\cdot)}\left[(1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)_+}\right]$

- Fix the input to $\mathcal{M}_1$ by conditioning Unif rand
- x := #a's given to $\mathcal{M}_1$ in the conditioned input
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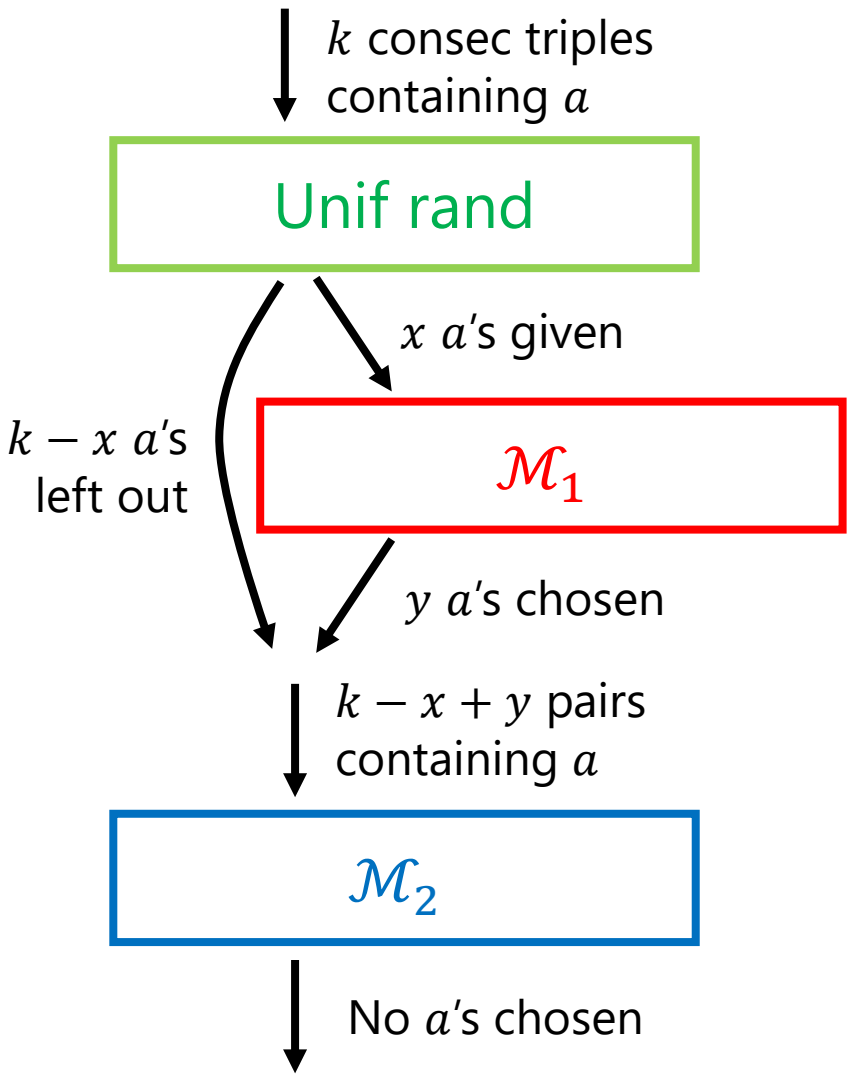
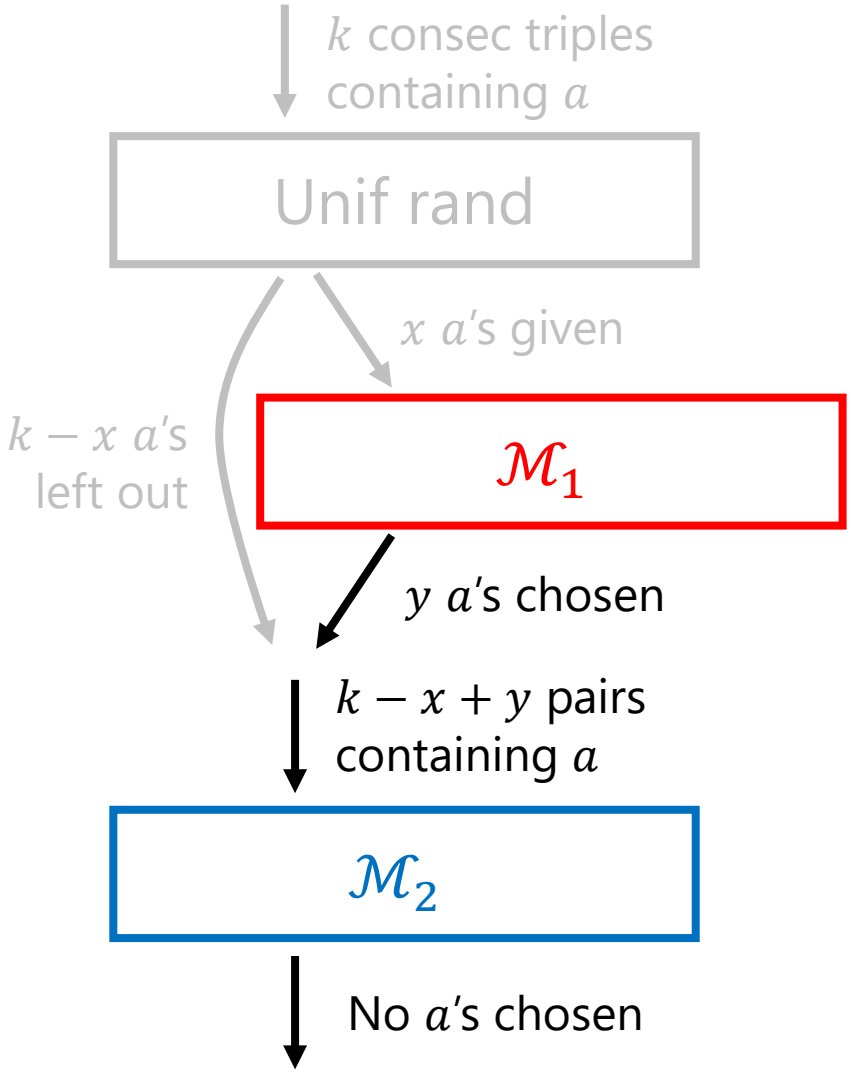
Property of Surrogate Distribution



- $P[a \text{ never chosen out of } k \text{ triples}]$
  $\leq \sum_{x=0}^{k} \mathrm{binom}(k, 2/3\,; x) \cdot$
  $\mathbb{E}_{y\sim p^*(x,\cdot)}\left[(1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)_+}\right]$
  $= \sum_{x=0}^{k} \mathrm{binom}(k, 2/3\,; x) \cdot$
  $\left[\sum_{y=0}^{x} p^*(x,y) \cdot (1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)_+}\right]$

- Recall that $p^*(x,y)$ depends only on x

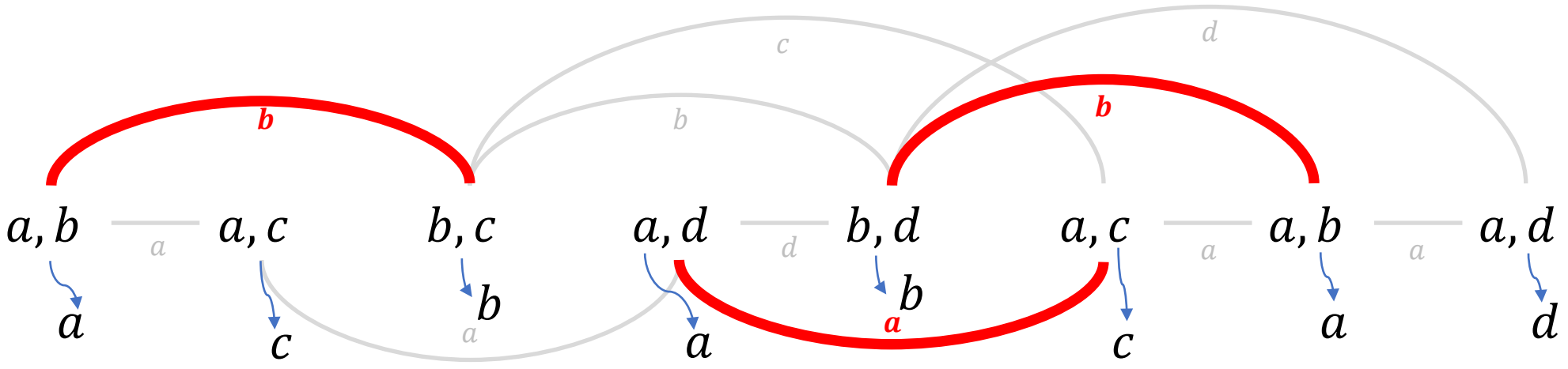
Outcome Distribution?



- $P[\, a$ never chosen out of k triples | Unif rand $]$
  $\leq \mathbb{E}_{y\sim p(\cdot)}\left[(1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)_+}\right]$
  $\leq \mathbb{E}_{y\sim p^*(x,\cdot)}\left[(1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)_+}\right]$
- x := #a's given to $\mathcal{M}_1$ in the conditioned input
- $p(y)$:= $P[y\ a'\text{s chosen by } \mathcal{M}_1$ | Unif rand$]$

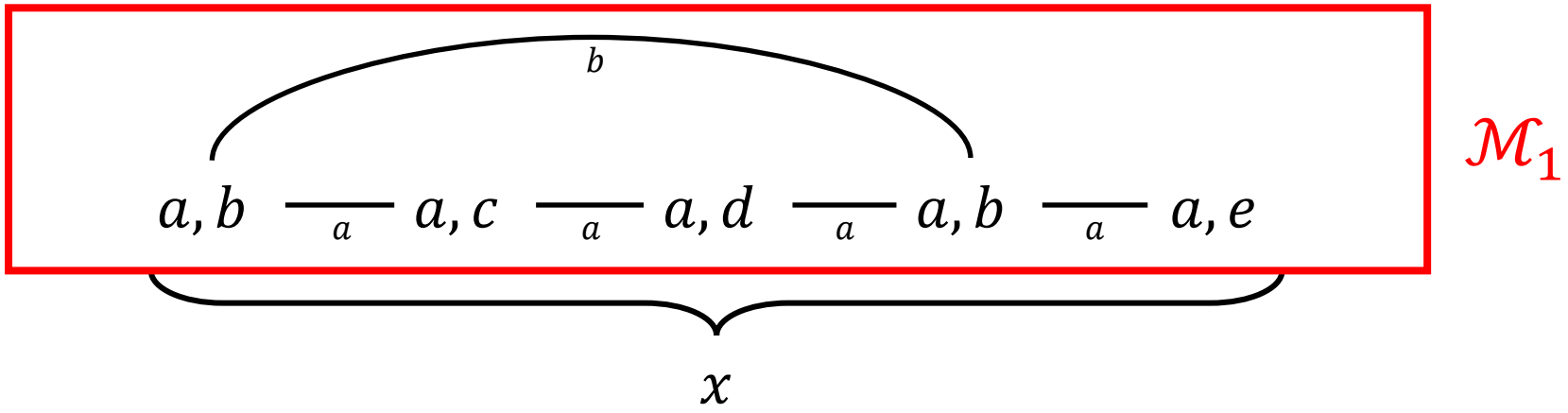
What does $p(y)$ look like?

Fahrbach et al.'s Two-Way OCS



- For each edge in the matching,
  - Choose one endpoint unif rand & output the common element
  - For the other endpoint, output the other element
- For every unmatched pair, choose one element unif rand

Outcome Distribution

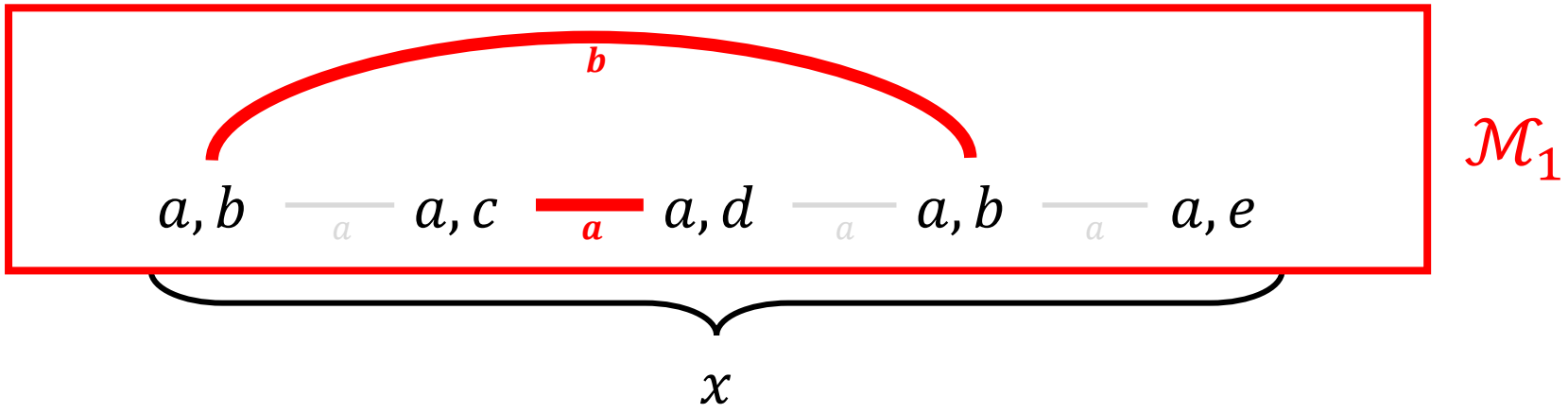


- If (#edges in the matching) $= 0$,

  $$(\#a\text{'s chosen by } \mathcal{M}_1) \sim \mathrm{binom}(x, 1/2)$$

- A unimodal symmetric distribution with mean $x/2$

Outcome Distribution

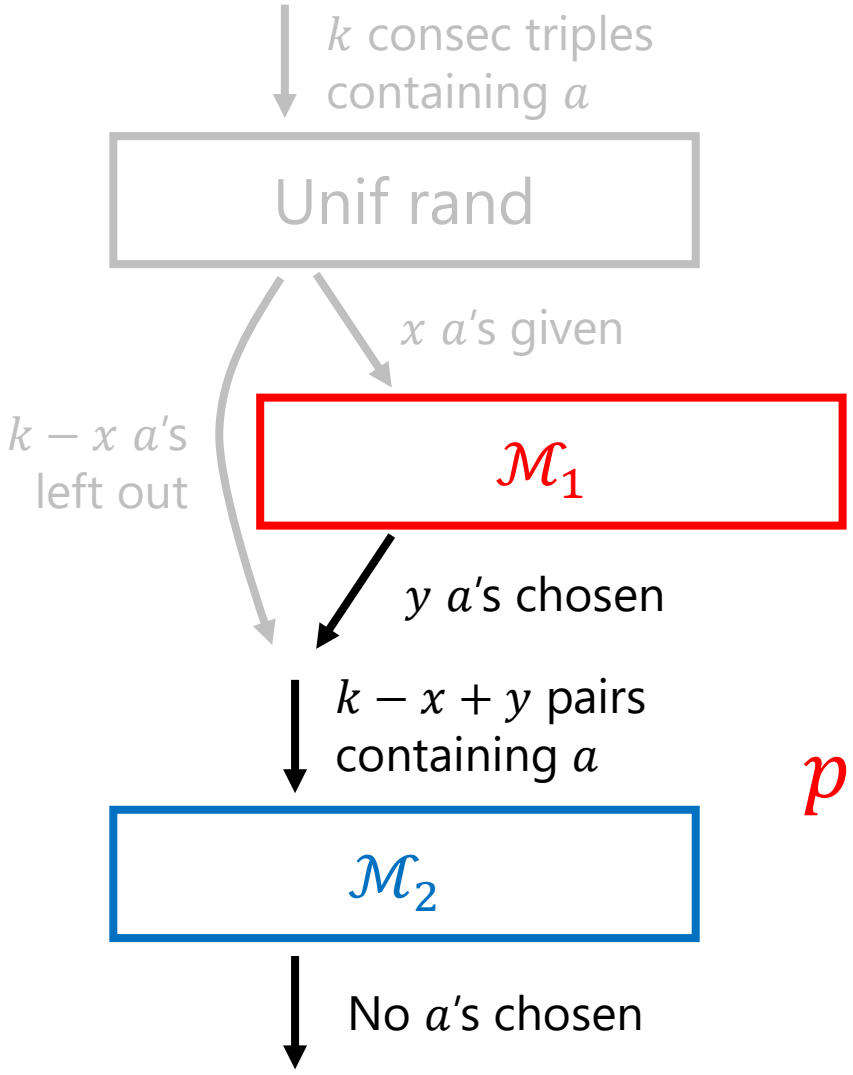


- If (#edges in the matching) $= 2$,

$$(\#a\text{'s chosen by } \mathcal{M}_1) \sim \mathrm{binom}(x-4, 1/2) + 2$$

- A unimodal symmetric distribution with mean $x/2$

Property of Surrogate Distribution

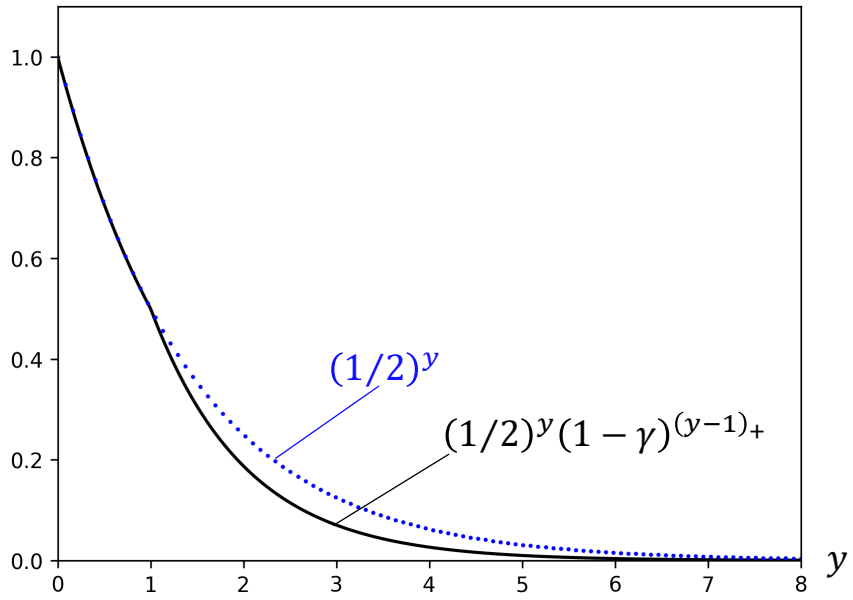


- $P[\, a$ never chosen out of k triples | Unif rand $]$
  $\leq \mathbb{E}_{y\sim p(\cdot)}\left[(1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)_+}\right]$
  $\leq \mathbb{E}_{y\sim p^*(x,\cdot)}\left[(1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)_+}\right]$
- x := #a's given to $\mathcal{M}_1$ in the conditioned input
- $p(y)$:= $P[y\ a'\text{s chosen by } \mathcal{M}_1$ | Unif rand$]$

$p(y)$ is a unimodal symmetric distribution!

Central Dominance

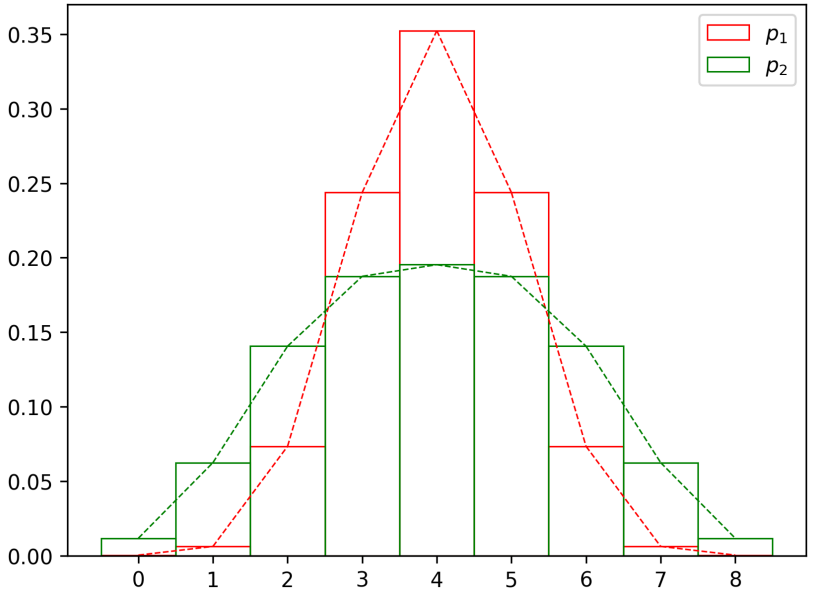
- Observe that $(1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)_+}$ is nearly convex

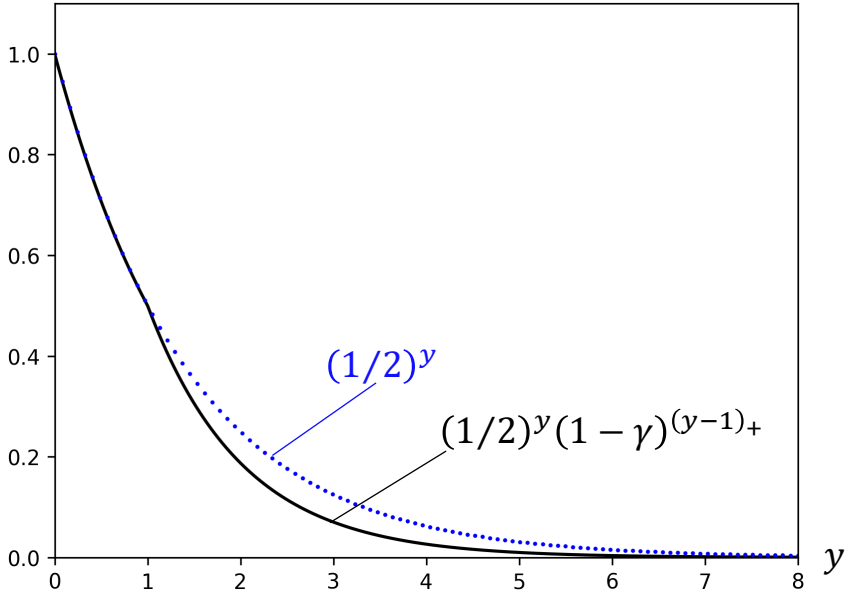


Central Dominance

- Observe that $(1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)_+}$ is nearly convex
- Given unimodal symmetric p_1, p_2 where p_2 is "**flatter**" than p_1,

$$\mathbb{E}_{y\sim p_1}\left[(1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)_+}\right] \leq \mathbb{E}_{y\sim p_2}\left[(1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)_+}\right]$$

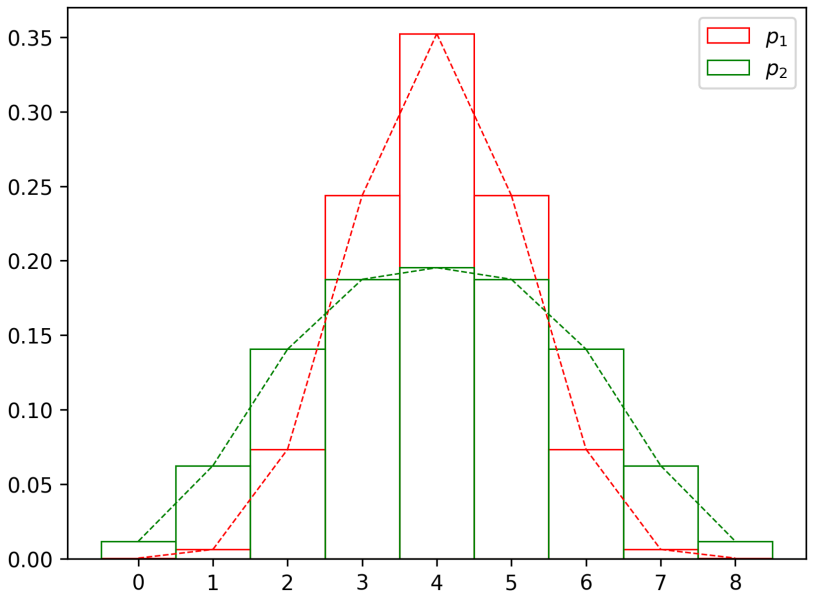


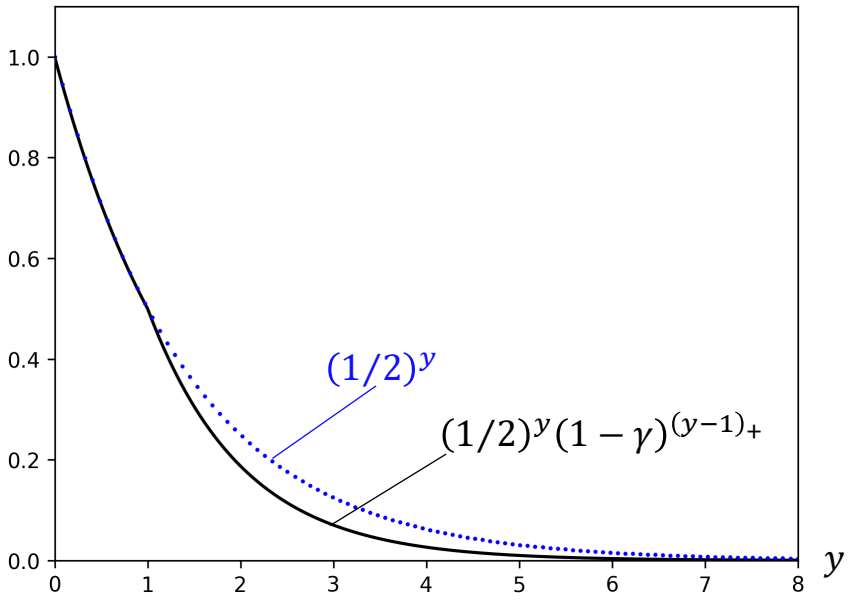


Central Dominance

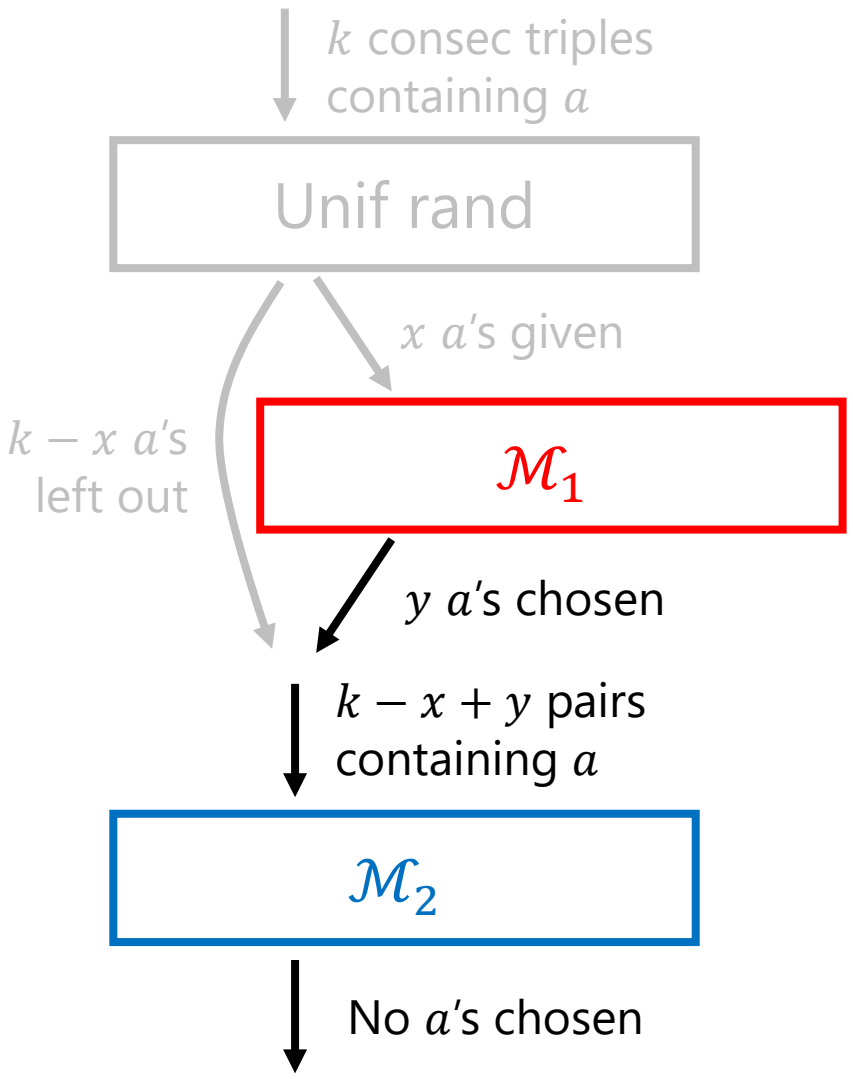
- Observe that $(1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)_+}$ is nearly convex
- Given unimodal symmetric p_1, p_2 where p_2 is **centrally dominated** by p_1,

$$\mathbb{E}_{y\sim p_1}\left[(1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)_+}\right] \leq \mathbb{E}_{y\sim p_2}\left[(1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)_+}\right]$$





Property of Surrogate Distribution

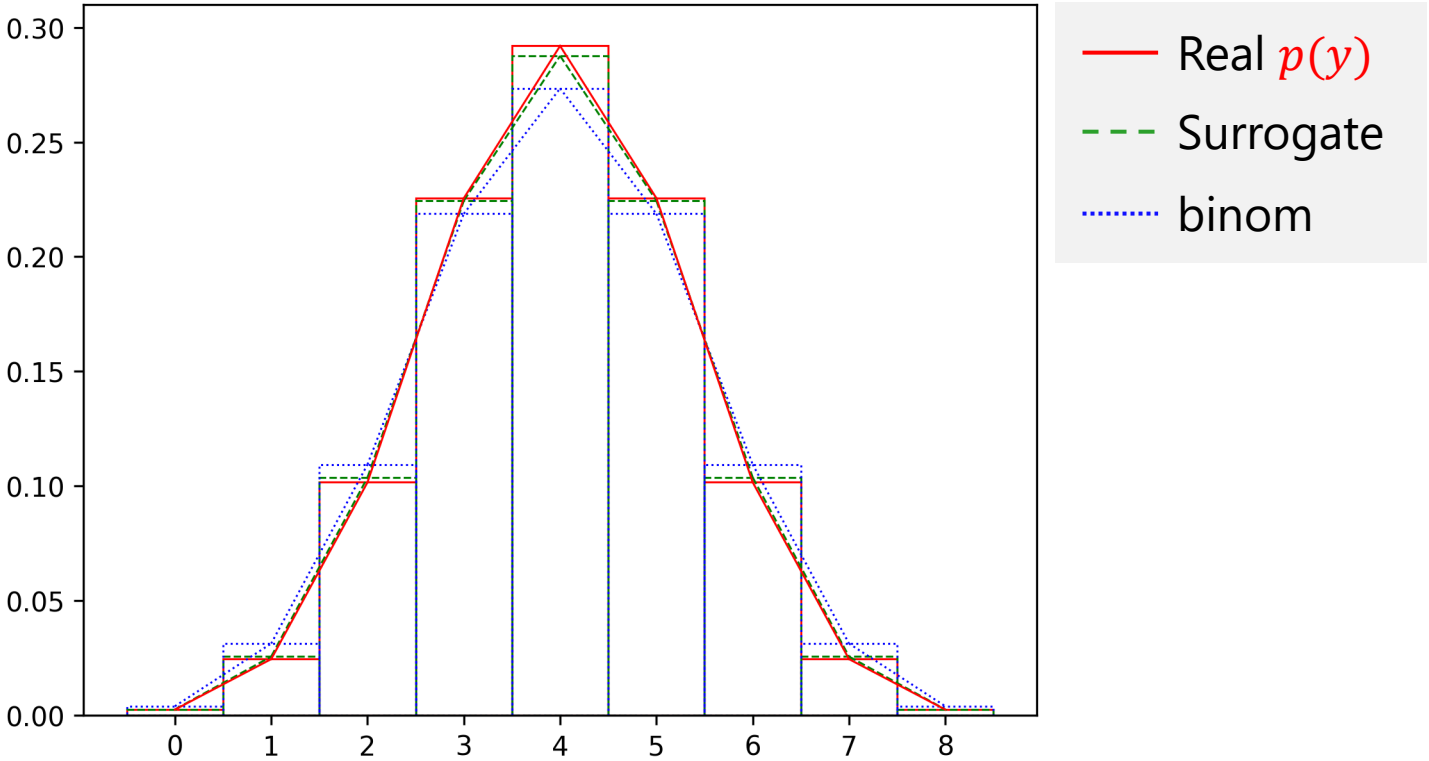


- $P[\, a$ never chosen out of k triples | Unif rand $]$
  $\leq \mathbb{E}_{y\sim p(\cdot)}\left[(1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)_+}\right]$
  $\leq \mathbb{E}_{y\sim p^*(x,\cdot)}\left[(1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)_+}\right]$
- $x := \#a$'s given to $\mathcal{M}_1$ in the conditioned input
- $p(y) := P[y\ a'\text{s chosen by } \mathcal{M}_1 \mid \text{Unif rand}]$

Want $p^*(x,y)$ to be centrally dominated by $p(y)$
(flatter than)

Devising a Good Centrally Dominated Dist

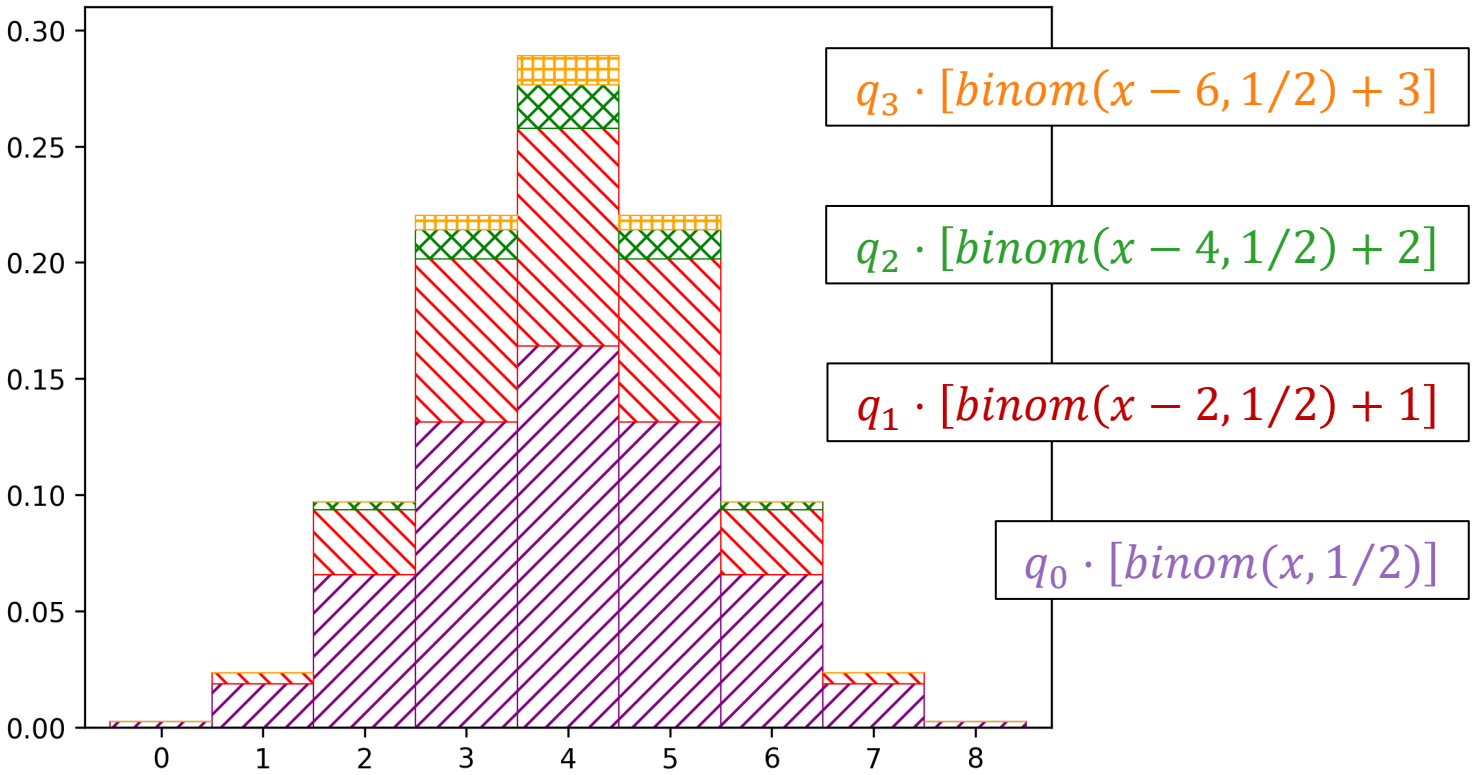
- $\mathrm{binom}(x, 1/2)$ is centrally dominated by any $p(y)$
  (flatter than)
- But need a pointier distribution for $p^*(x, y)$



Distribution on #a's chosen by $\mathcal{M}_1$

Devising a Good Centrally Dominated Dist

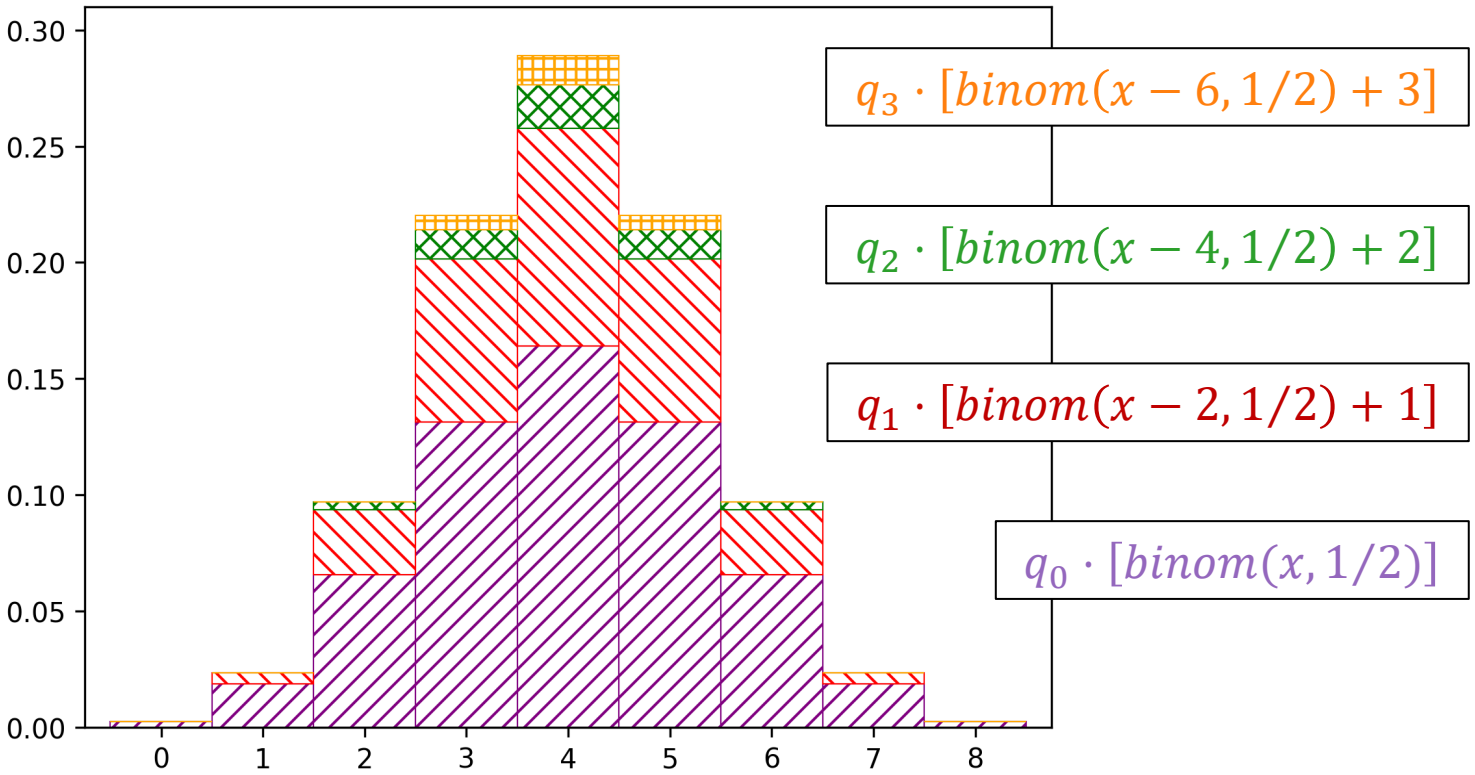
- Let q_i be $P[(\#\text{edges in the matching}) = i]$ for $i = 0, \cdots, \lfloor x/2 \rfloor$



Distribution on $\#a$'s chosen by $\mathcal{M}_1$

Devising a Good Centrally Dominated Dist

- Let q_i be $P[(\#\text{edges in the matching}) = i]$ for $i = 0, \cdots, \lfloor x/2 \rfloor$
- $q_0 \leq (1 - 1/16)^{(x-1)_+} =: \alpha_x$ for any $p(y)$

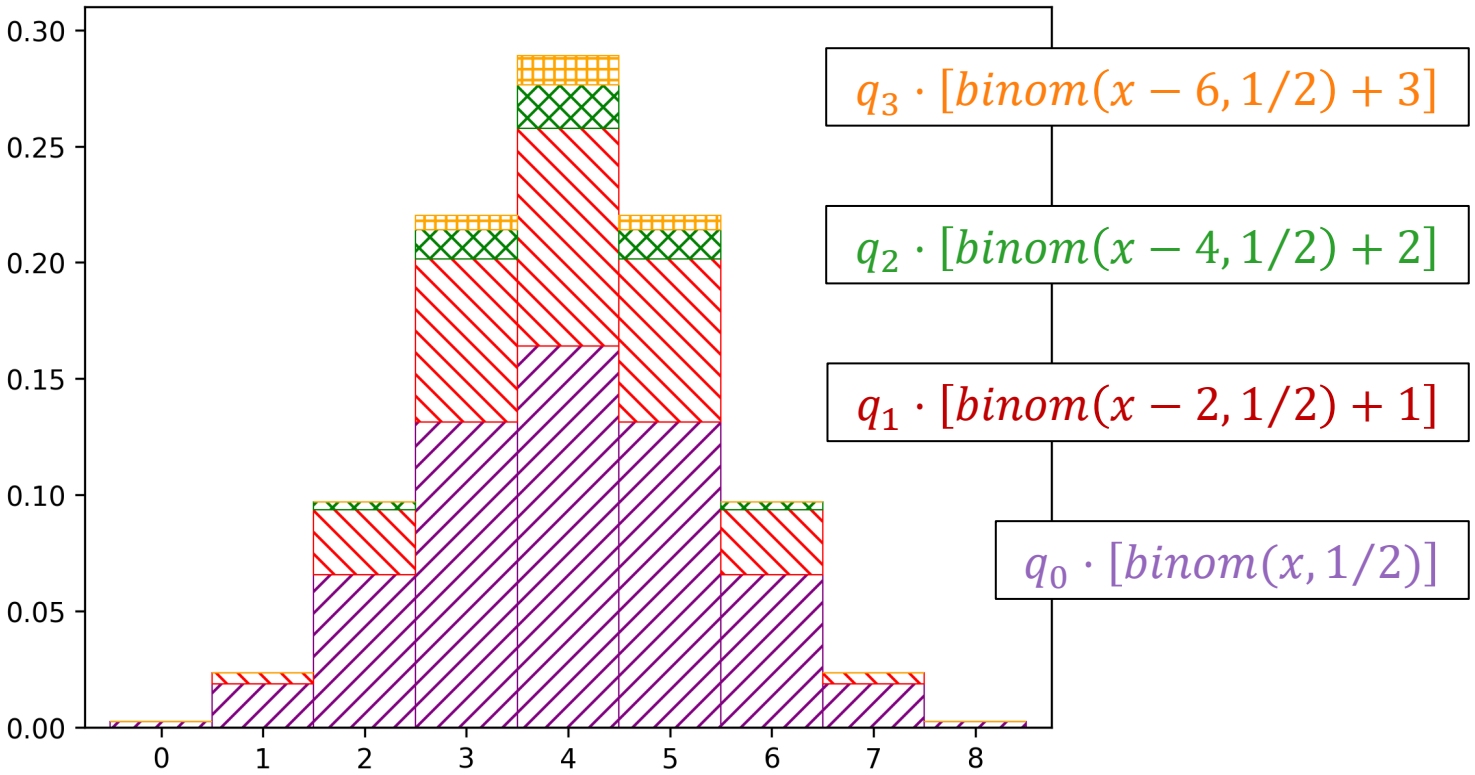


Distribution on #a's chosen by $\mathcal{M}_1$

q_3	?		0
q_2	?		0
q_1	?		0
q_0	$\leq \alpha_x$		1
	$p(y)$		$\text{binom}(x, \frac{1}{2})$

Devising a Good Centrally Dominated Dist

- Let q_i be $P[(\#\text{edges in the matching}) = i]$ for $i = 0, \cdots, \lfloor x/2 \rfloor$
- $q_0 \leq (1 - 1/16)^{(x-1)_+} =: \alpha_x$ for any $p(y)$

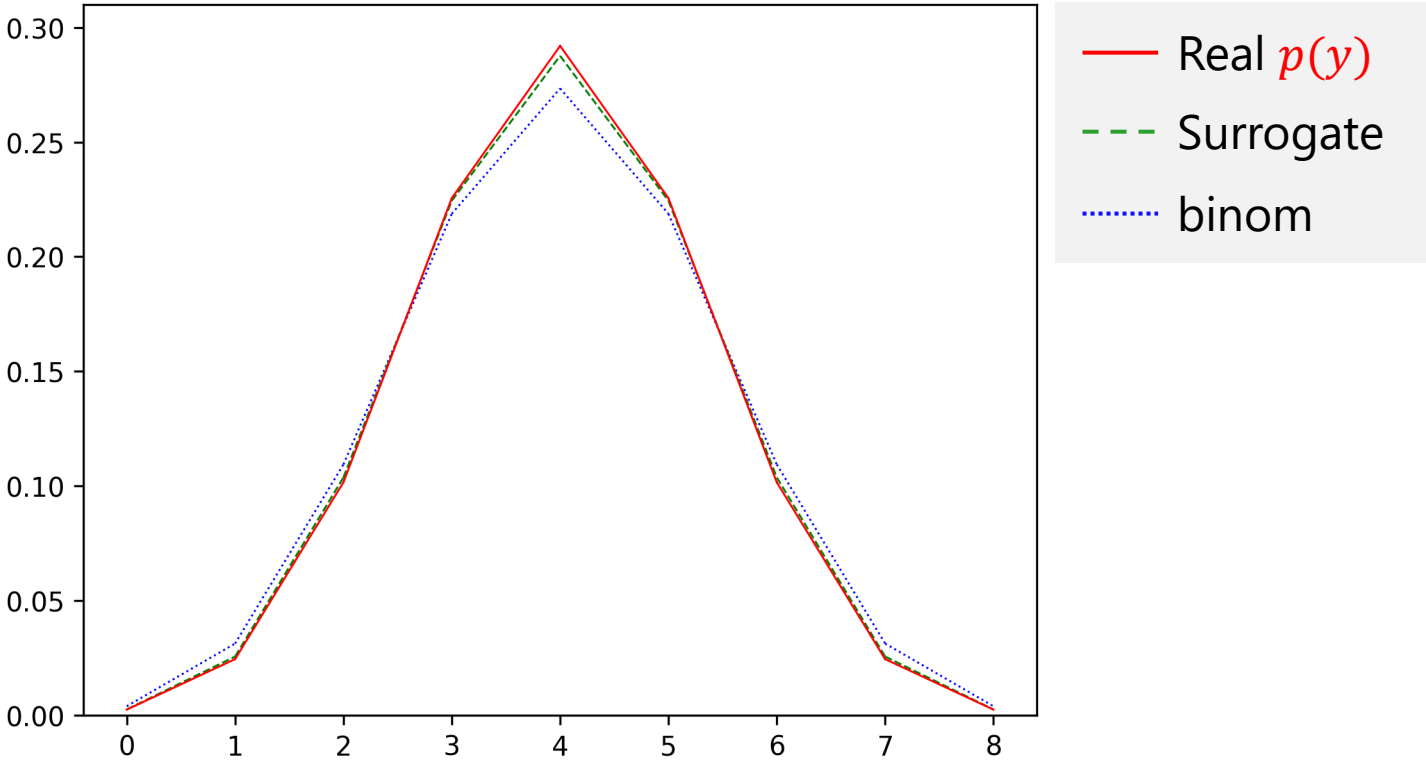


Distribution on #a's chosen by $\mathcal{M}_1$

q_3	?	0	0
q_2	?	0	0
q_1	?	$1 - \alpha_x$	0
q_0	$\leq \alpha_x$	α_x	1
	$p(y)$	$p^*(x, y)$	$\text{binom}(x, \frac{1}{2})$

Devising a Good Centrally Dominated Dist

- Let q_i be $P[(\#\text{edges in the matching}) = i]$ for $i = 0, \cdots, \lfloor x/2 \rfloor$
- $q_0 \leq (1 - 1/16)^{(x-1)_+} =: \alpha_x$ for any $p(y)$



Distribution on $\#a$'s chosen by $\mathcal{M}_1$

(Final) Calculation

- $P[a \text{ never chosen from a single subseq of length } k]$

  $\leq \sum_{x=0}^{k} \text{binom}\left(k, \frac{2}{3}; x\right) \times \left[\sum_{y=0}^{x} p^*(x,y)\,(1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)_+}\right]$

  $= c_1 t_1^k + c_2 t_2^k - c_3 t_3^k - c_4 t_4^k$

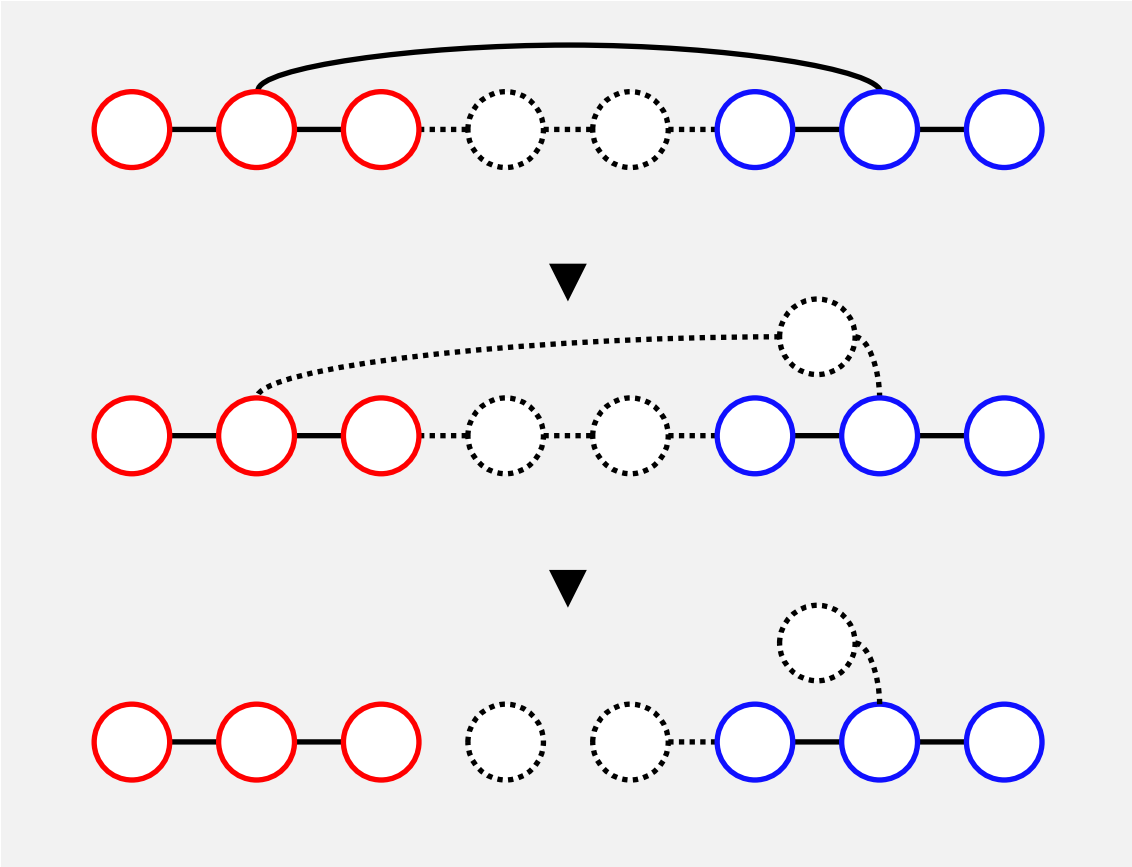
  $\leq \left(\frac{2}{3}\right)^k (1-\delta_1)^{(k-1)_+}(1-\delta_2)^{(k-2)_+}$

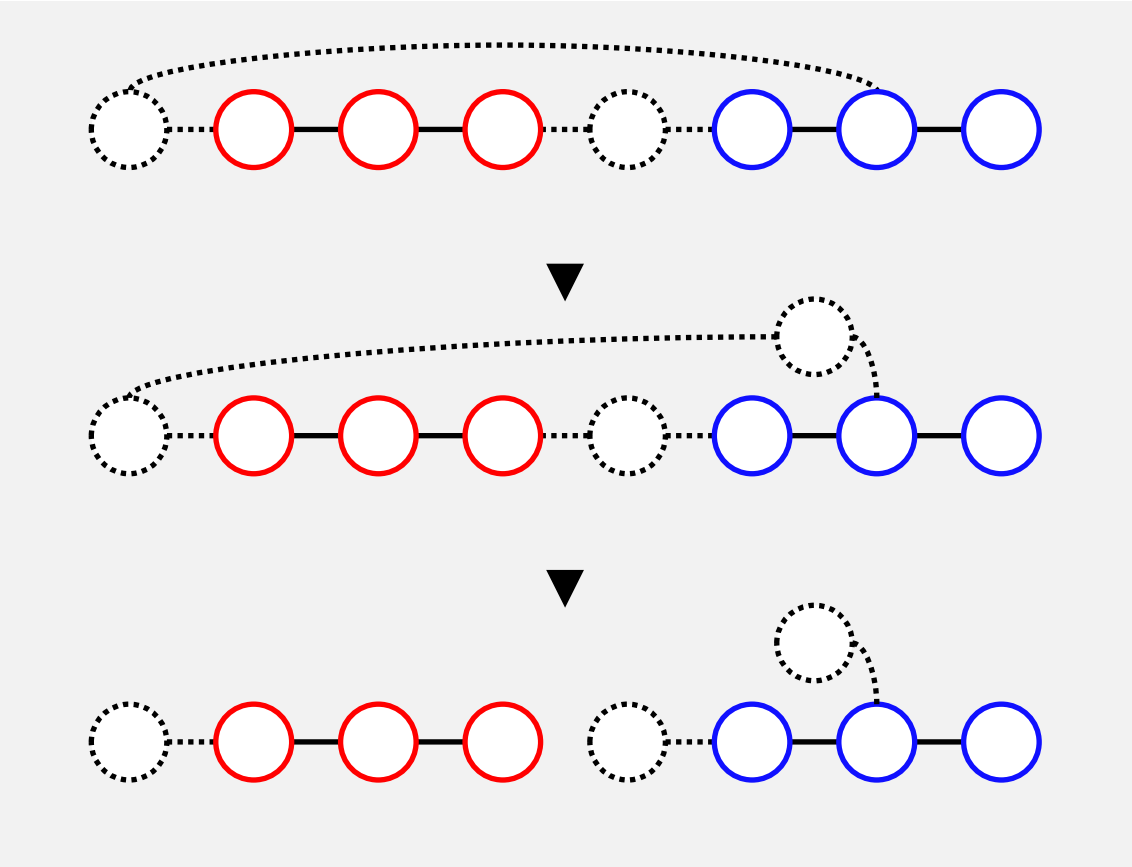
- $c_1 \approx 0.95, c_2 \approx 0.17, c_3 \approx 0.01, c_4 \approx 0.13$
- $t_1 \approx 0.63, t_2 \approx 0.59, t_3 \approx 0.14, t_4 \approx 0.31$
- $\delta_1 \approx 0.03, \delta_2 \approx 0.01$

Extending to General Case

- $P[a \text{ never chosen from subseqs of lengths } k_1, \cdots, k_\ell]$
  $\leq \prod_{i=1}^{\ell} c_1 t_1^{k_i} + c_2 t_2^{k_i} - c_3 t_3^{k_i} - c_4 t_4^{k_i}$
  $\leq \prod_{i=1}^{\ell} \left(\frac{2}{3}\right)^{k_i} (1-\delta_1)^{(k_i-1)_+} (1-\delta_2)^{(k_i-2)_+}$
- Need to remove correlations between subseqs in $\mathcal{M}_1$
- Can remove correlations of 1/16-OCS by surgical operations

Surgical Operations





Applications

- With our three-way OCS,
  - a 0.5096-competitive alg for unweighted matching
  - a 0.5093-competitive alg for edge-weighted matching

Future Directions

- Generalization to >3-way OCS
  - Will a “cascaded” OCS work?
- Other applications of OCS
  - Negative correlation

Thank you for your attention