Making Three out of Two: Three-Way Online Correlated Selection

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Two-Way Online Correlated Selection (OCS)

- Given a sequence of pairs arriving one-by-one at each timestep, choose an element from each pair irrevocably

- **Definition** (Two-way $\gamma$-OCS) [Fahrbach et al.]
  - For any element $a$ and a set of disjoint consecutive subsequences containing $a$ of length $k_1, \ldots, k_\ell$,
  
  $$P[a \text{ never chosen from subseqs}] \leq \prod_{i=1}^{\ell} \frac{(1/2)^{k_i}(1 - \gamma)^{(k_i-1)_+}}{1 - \gamma}$$

- **Negative Correlation**

![Diagram of negative correlation]
Two-Way Online Correlated Selection (OCS)

• Invented by Fahrbach, Huang, Tao, Zadimoghaddam (FOCS 2020) to tackle the edge-weighted online bipartite matching problem

• Introduces negative correlation to online algorithms

• Fahrbach et al. present

  • 1/16-OCS \( \leq \prod_{i=1}^{\ell} (1/2)^{k_i} (1 - 1/16)^{(k_{i-1})+} \)

  • 0.1099-OCS \( \leq \prod_{i=1}^{\ell} (1/2)^{k_i} (1 - 0.1099)^{(k_{i-1})+} \)
Open Question

• Can we obtain a nontrivial >2-way OCS?

  Yes. We give a three-way OCS.
Independent Work

• Blanc and Charikar (FOCS 2021)
  • \((F,m)\)-OCS
  • Continuous-OCS

• Gao et al. (FOCS 2021)
  • Automata-based two-way 0.167-OCS
  • Multiway semi-OCS
Main Result: Three-Way OCS

• Given a sequence of **triples** arriving one-by-one at each time, choose an element from each **triple** irrevocably

• For any element \(a\) and a set of disjoint consecutive subseqs containing \(a\) of length \(k_1, \ldots, k_\ell\),

\[
P[a \text{ never chosen from subseqs}] \leq \prod_{i=1}^{\ell} \left(\frac{2}{3}\right)^{k_i} (1 - \delta_1)^{(k_i-1)+} (1 - \delta_2)^{(k_i-2)+}
\]
Our Algorithm

Unif rand

A two-way OCS $\mathcal{M}_1$

A two-way OCS $\mathcal{M}_2$
Special Case: A Single Subsequence

- $P[a$ never chosen out of $k$ triples$] \\ \leq \sum_{x=0}^{k} P[x$ a's given to $\mathcal{M}_1$ out of $k] \cdot \left[ \sum_{y=0}^{x} P[y$ a's chosen by $\mathcal{M}_1$ out of $x] \cdot P[No a's chosen by $\mathcal{M}_2$ out of $k-x+y] \right]$
Special Case: A Single Subsequence

\[ P[a \text{ never chosen out of } k \text{ triples}] \leq \sum_{x=0}^{k} \binom{k}{x} \cdot \frac{2/3}{x} \cdot \sum_{y=0}^{x} P[y \text{ a's chosen by } M_1 \text{ out of } x] \cdot \left( \frac{1}{2} \right)^{k-x+y} (1 - \gamma)^{(k-x+y-1)_+} \]
Major Difficulty in Analysis

- $P[a\ never\ chosen\ out\ of\ k\ triples]\leq\sum_{x=0}^{k}\binom{k}{2/3;x}\cdot[\sum_{y=0}^{x}P[y\ a's\ chosen\ by\ M_1\ out\ of\ x] \cdot (1/2)^{k-x+y}(1-\gamma)^{(k-x+y-1)+}]$

- $P[y\ a's\ chosen\ by\ M_1\ out\ of\ x]$
  - was not studied by previous analyses
  - depends on the actual input, not on $x$
  - does not have a closed-form formula
Main Idea of Analysis

- $P[a$ never chosen out of $k$ triples$] \leq \sum_{x=0}^{k} \binom{k}{x} \cdot \left[ \sum_{y=0}^{x} \binom{x}{y} \cdot \frac{1}{2}^{k-x+y} \cdot (1 - \gamma)^{(k-x+y-1)+} \right]$

- Construct a “surrogate” distribution $p^*(x, y)$
  - depending only on $x$
  - yielding a closed-form formula
Fixing the Input to $\mathcal{M}_1$

- $P[ a \text{ never chosen out of } k \text{ triples } | \text{ Unif rand } ]$
  \[ \leq \sum_{y=0}^{x} p(y) \times (1/2)^{k-x+y} (1 - \gamma)^{(k-x+y-1)_+} \]

- Fix the input to $\mathcal{M}_1$ by conditioning Unif rand
- $x := \#a's \text{ given to } \mathcal{M}_1 \text{ in the conditioned input}$
- $p(y) := P[y a's \text{ chosen by } \mathcal{M}_1 | \text{ Unif rand}]$
Property of Surrogate Distribution

- \( P[ a \text{ never chosen out of } k \text{ triples } | \text{ Unif rand } ] \)
  \[ \leq \mathbb{E}_{y \sim p(\cdot)}[(1/2)^{k-x+y} (1 - y)^{(k-x+y-1)+}] \]
  \[ \leq \mathbb{E}_{y \sim p^*(x, \cdot)}[(1/2)^{k-x+y} (1 - y)^{(k-x+y-1)+}] \]

- Fix the input to \( \mathcal{M}_1 \) by conditioning \( \text{Unif rand} \)
- \( x := \#a's \text{ given to } \mathcal{M}_1 \) in the conditioned input
- \( p(y) := P[y a's \text{ chosen by } \mathcal{M}_1 | \text{ Unif rand}] \)
Property of Surrogate Distribution

\[ P[a \text{ never chosen out of } k \text{ triples}] \leq \sum_{x=0}^{k} \binom{k}{2/3} x \cdot \mathbb{E}_{y \sim p^*(x, \cdot)}[(1/2)^{k-x+y}(1 - \gamma)^{(k-x+y-1)+}] \]

\[ = \sum_{x=0}^{k} \binom{k}{2/3} x \cdot \left[ \sum_{y=0}^{x} p^*(x, y) \cdot (1/2)^{k-x+y}(1 - \gamma)^{(k-x+y-1)+} \right] \]

- Recall that \( p^*(x, y) \) depends only on \( x \)
Outcome Distribution?

- $P[ a \text{ never chosen out of } k \text{ triples | Unif rand }]$
  \leq \mathbb{E}_{y \sim p(\cdot)}[(1/2)^{k-x+y}(1 - y)^{(k-x+y-1)+}]$
  \leq \mathbb{E}_{y \sim p^*(x, \cdot)}[(1/2)^{k-x+y}(1 - y)^{(k-x+y-1)+}]$

- $x := \#a's \text{ given to } \mathcal{M}_1 \text{ in the conditioned input}$
- $p(y) := P[y a's \text{ chosen by } \mathcal{M}_1 | \text{ Unif rand}]$

What does $p(y)$ look like?
Fahrbach et al.'s Two-Way OCS

- For each edge in the matching,
  - Choose one endpoint unif rand & output the common element
  - For the other endpoint, output the other element
- For every unmatched pair, choose one element unif rand
Outcome Distribution

- If (#edges in the matching) = 0,
  
  (#$a$’s chosen by $\mathcal{M}_1$) $\sim$ binom($x$, 1/2)

- A unimodal symmetric distribution with mean $x/2$
Outcome Distribution

• If (number of edges in the matching) = 2,
  \((\#a's\ chosen\ by\ \mathcal{M}_1) \sim \text{binom}(x-4, 1/2) + 2\)

• A unimodal symmetric distribution with mean \(x/2\)
Property of Surrogate Distribution

- \( P[ \text{a never chosen out of } k \text{ triples} \mid \text{Unif rand} ] \)
- \( \leq \mathbb{E}_{y \sim p(\cdot)}[(1/2)^{k-x+y}(1 - y)^{(k-x+y-1)+}] \)
- \( \leq \mathbb{E}_{y \sim p^*(x, \cdot)}[(1/2)^{k-x+y}(1 - y)^{(k-x+y-1)+}] \)

- \( x := \#a's \text{ given to } \mathcal{M}_1 \text{ in the conditioned input} \)
- \( p(y) := P[y \text{ a's chosen by } \mathcal{M}_1 \mid \text{Unif rand}] \)

\( p(y) \) is a unimodal symmetric distribution!
Central Dominance

• Observe that \((1/2)^{k-x+y} (1 - \gamma)^{(k-x+y-1)_+}\) is nearly convex
Central Dominance

• Observe that \((1/2)^{k-x+y} (1 - \gamma)^{(k-x+y-1)+}\) is nearly convex

• Given unimodal symmetric \(p_1, p_2\) where \(p_2\) is "flatter" than \(p_1\),
  \[
  \mathbb{E}_{y \sim p_1} [(1/2)^{k-x+y} (1 - \gamma)^{(k-x+y-1)+}] \leq \mathbb{E}_{y \sim p_2} [(1/2)^{k-x+y} (1 - \gamma)^{(k-x+y-1)+}]
  \]
Central Dominance

• Observe that \((1/2)^{k-x+y} (1 - \gamma)^{(k-x+y-1)_+}\) is nearly convex

• Given unimodal symmetric \(p_1, p_2\) where \(p_2\) is centrally dominated by \(p_1\),

\[
\mathbb{E}_{y \sim p_1} \left[ (1/2)^{k-x+y} (1 - \gamma)^{(k-x+y-1)_+} \right] \leq \mathbb{E}_{y \sim p_2} \left[ (1/2)^{k-x+y} (1 - \gamma)^{(k-x+y-1)_+} \right]
\]
Property of Surrogate Distribution

- $P[\ a \ never \ chosen \ out \ of \ k \ triples \ | \ \text{Unif rand}]$
  \leq \mathbb{E}_{y \sim p(\cdot)}[(1/2)^{k-x+y}(1-y)^{(k-x+y-1)+}]$
  \leq \mathbb{E}_{y \sim p^*(x,\cdot)}[(1/2)^{k-x+y}(1-y)^{(k-x+y-1)+}]$

- $x := \#a's \ given \ to \ M_1 \ in \ the \ conditioned \ input$
- $p(y) := P[y \ a's \ chosen \ by \ M_1 | \ \text{Unif rand}]$

Want $p^*(x, y)$ to be centrally dominated by $p(y)$ (flatter than)
Devising a Good Centrally Dominated Dist

- $\text{binom}(x, 1/2)$ is centrally dominated by any $p(y)$ (flatter than)
- But need a pointier distribution for $p^*(x, y)$
Devising a Good Centrally Dominated Dist

• Let \( q_i \) be \( P[(\#\text{edges in the matching}) = i] \) for \( i = 0, \ldots, \lfloor x/2 \rfloor \)
Devising a Good Centrally Dominated Dist

• Let $q_i$ be $P[(\#edges\ in\ the\ matching) = i]$ for $i = 0, \cdots, \lfloor x/2 \rfloor$

• $q_0 \leq (1 - 1/16)^{(x-1)+} =: \alpha_x$ for any $p(y)$

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<thead>
<tr>
<th>$q_0$</th>
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$p(y)$:

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Distribution on $\#a's$ chosen by $\mathcal{M}_1$
Devising a Good Centrally Dominated Dist

• Let $q_i$ be $P[(\#\text{edges in the matching}) = i]$ for $i = 0, \cdots, [x/2]$.

• $q_0 \leq (1 - 1/16)^{(x-1)+} =: \alpha_x$ for any $p(y)$.

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<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$p(y)$ | $p^*(x,y)$ | $\text{binom}(x, \frac{1}{2})$ |
Devising a Good Centrally Dominated Dist

- Let $q_i$ be $P[(\#\text{edges in the matching}) = i]$ for $i = 0, \cdots, \lfloor x/2 \rfloor$
- $q_0 \leq (1 - 1/16)^{(x-1)+} =: \alpha_x$ for any $p(y)$
(Final) Calculation

• \( P[a \text{ never chosen from a single subseq of length } k] \)
  \[
  \leq \sum_{x=0}^{k} \binom{k}{x} \frac{2}{3}^x \times \left[ \sum_{y=0}^{x} p^*(x, y) \left( \frac{1}{2} \right)^{k-x+y} (1 - y)^{(k-x+y-1)+} \right]
  \]
  \[
  = c_1 t_1^k + c_2 t_2^k - c_3 t_3^k - c_4 t_4^k
  \]
  \[
  \leq \left( \frac{2}{3} \right)^k (1 - \delta_1)^{(k-1)+} (1 - \delta_2)^{(k-2)+}
  \]

• \( c_1 \approx 0.95, c_2 \approx 0.17, c_3 \approx 0.01, c_4 \approx 0.13 \)

• \( t_1 \approx 0.63, t_2 \approx 0.59, t_3 \approx 0.14, t_4 \approx 0.31 \)

• \( \delta_1 \approx 0.03, \delta_2 \approx 0.01 \)
Extending to General Case

• $P[a$ never chosen from subseqs of lengths $k_1, \ldots, k_\ell]$ 

$$\leq \prod_{i=1}^{\ell} c_1 t_1^{k_i} + c_2 t_2^{k_i} - c_3 t_3^{k_i} - c_4 t_4^{k_i}$$

$$\leq \prod_{i=1}^{\ell} \left(\frac{2}{3}\right)^{k_i} (1 - \delta_1)^{(k_i - 1)} + (1 - \delta_2)^{(k_i - 2)}$$

• Need to remove correlations between subseqs in $\mathcal{M}_1$

• Can remove correlations of 1/16-OCS by surgical operations
Surgical Operations
Applications

• With our three-way OCS,
  • a 0.5096-competitive alg for unweighted matching
  • a 0.5093-competitive alg for edge-weighted matching
Future Directions

• Generalization to >3-way OCS
  • Will a “cascaded” OCS work?
• Other applications of OCS
  • Negative correlation
Thank you for your attention